

# ON THE VALUE SET OF SMALL FAMILIES OF POLYNOMIALS OVER A FINITE FIELD, III

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**ABSTRACT.** We estimate the average cardinality  $\mathcal{V}(\mathcal{A})$  of the value set of a general family  $\mathcal{A}$  of monic univariate polynomials of degree  $d$  with coefficients in the finite field  $\mathbb{F}_q$ . We establish conditions on the family  $\mathcal{A}$  under which  $\mathcal{V}(\mathcal{A}) = \mu_d q + \mathcal{O}(q^{1/2})$ , where  $\mu_d := \sum_{r=1}^d (-1)^{r-1}/r!$ . The result holds without any restriction on the characteristic of  $\mathbb{F}_q$  and provides an explicit expression for the constant underlying the  $\mathcal{O}$ -notation in terms of  $d$ . We reduce the question to estimating the number of  $\mathbb{F}_q$ -rational points with pairwise-distinct coordinates of a certain family of complete intersections defined over  $\mathbb{F}_q$ . For this purpose, we obtain an upper bound on the dimension of the singular locus of the complete intersections under consideration, which allows us to estimate the corresponding number of  $\mathbb{F}_q$ -rational points.

## 1. INTRODUCTION

Let  $\mathbb{F}_q$  be the finite field of  $q := p^s$  elements, where  $p$  is a prime number, let  $\overline{\mathbb{F}}_q$  denote its algebraic closure, and let  $T$  be an indeterminate over  $\overline{\mathbb{F}}_q$ . For  $f \in \mathbb{F}_q[T]$ , its value set is the image of the mapping from  $\mathbb{F}_q$  to  $\mathbb{F}_q$  defined by  $f$  (cf. [LN83]). We shall denote its cardinality by  $\mathcal{V}(f)$ , namely  $\mathcal{V}(f) := |\{f(c) : c \in \mathbb{F}_q\}|$ .

In a seminal paper, Birch and Swinnerton-Dyer [BS59] showed that, for fixed  $d \geq 1$ , if  $f \in \mathbb{F}_q[T]$  is a “general” polynomial of degree  $d$ , then

$$(1.1) \quad \mathcal{V}(f) = \mu_d q + \mathcal{O}(q^{\frac{1}{2}}),$$

where  $\mu_d := \sum_{r=1}^d (-1)^{r-1}/r!$  and the  $\mathcal{O}$ -constant depends only on  $d$ .

Uchiyama [Uch55] and Cohen ([Coh73], [Coh72]) were concerned on estimates for the average cardinality of the value set when  $f$  ranges over all monic polynomials of degree  $d$  in  $\mathbb{F}_q[T]$ . In particular, in [Coh72] the problem of estimating the average cardinality of the value set on linear families of monic polynomials of  $\mathbb{F}_q[T]$  of degree  $d$  is addressed. More precisely, it is shown that, for a linear family  $\mathcal{A}$  of codimension  $m \leq d - 2$  satisfying certain conditions,

$$(1.2) \quad \mathcal{V}(\mathcal{A}) = \mu_d q + \mathcal{O}(q^{\frac{1}{2}}),$$

where  $\mathcal{V}(\mathcal{A})$  denotes the average cardinality of the value set of the elements in  $\mathcal{A}$ . As a particular case we have the classical case of polynomials with prescribed coefficients, where simpler conditions are obtained.

A difficulty with (1.2) is that the hypotheses on the linear family  $\mathcal{A}$  seem complicated and not easy to verify. A second concern is that (1.2) imposes the restriction  $p > d$ , which inhibits its application to fields of small characteristic. For these reasons, in [CMPP14] and [MPP14] we obtained explicit estimates for any family of monic polynomials of  $\mathbb{F}_q[T]$  of degree  $d$  with certain consecutive coefficients prescribed, which are valid for  $p > 2$ . In this paper we develop a framework which allows us to significantly generalize these results to rather general (eventually nonlinear) families of monic polynomials of  $\mathbb{F}_q[T]$  of degree  $d$ .

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More precisely, let  $d, m$  be positive integers with  $q > d \geq m + 2$ , and let  $A_{d-1}, \dots, A_0$  be indeterminates over  $\overline{\mathbb{F}}_q$ . Let  $G_1, \dots, G_m \in \mathbb{F}_q[A_{d-1}, \dots, A_0]$  be polynomials of degree  $d_1, \dots, d_m$  and  $\mathcal{A} := \mathcal{A}(G_1, \dots, G_m)$  the family

$$(1.3) \quad \mathcal{A} := \left\{ T^d + \sum_{j=0}^{d-1} a_j T^j \in \mathbb{F}_q[T] : G_i(a_{d-1}, \dots, a_0) = 0 \ (1 \leq i \leq m) \right\}.$$

Denote by  $\mathcal{V}(\mathcal{A})$  the average value of  $\mathcal{V}(f)$  for  $f$  ranging in  $\mathcal{A}$ , that is,

$$(1.4) \quad \mathcal{V}(\mathcal{A}) := \frac{1}{|\mathcal{A}|} \sum_{f \in \mathcal{A}} \mathcal{V}(f).$$

Our main result establishes rather general conditions on  $G_1, \dots, G_m$  under which the asymptotic behavior of  $\mathcal{V}(\mathcal{A})$  agrees with that of the general case, as predicted in (1.1) and (1.2). More precisely, we prove that

$$|\mathcal{V}(\mathcal{A}) - \mu_d q| \leq 2^d \delta (3D + d^2) q^{\frac{1}{2}} + 67 \delta^2 (D + 2)^2 d^{d+5} e^{2\sqrt{d}-d},$$

where  $\delta := \prod_{i=1}^m d_i$  and  $D := \sum_{i=1}^m (d_i - 1)$ .

Our approach relies on tools of algebraic geometry in the same vein as [CMPP14] and [MPP14]. In Section 2 we recall the basic notions and notations of algebraic geometry we use. In Section 3 we provide a combinatorial expression for  $\mathcal{V}(\mathcal{A})$  in terms of the number  $\mathcal{S}_r^{\mathcal{A}}$  of certain “interpolating sets” with  $1 \leq r \leq d$  and we relate each  $\mathcal{S}_r^{\mathcal{A}}$  with the number of  $\mathbb{F}_q$ -rational points of certain incidence variety  $\Gamma_r^*$  of  $\overline{\mathbb{F}}_q^{d+r}$ . In Section 4 we show that  $\Gamma_r^*$  is an  $\mathbb{F}_q$ -definable normal complete intersection, and establish a number of geometric properties of  $\Gamma_r^*$ . To estimate the number of  $\mathbb{F}_q$ -rational points of  $\Gamma_r^*$  is necessary to discuss the behavior of  $\Gamma_r^*$  at “infinity”, which is done in Section 5. Finally, the results of Sections 4 and 5 allow us to estimate, in Section 6, the number of  $\mathbb{F}_q$ -rational points of  $\Gamma_r^*$ , and therefore determine the asymptotic behavior of  $\mathcal{V}(\mathcal{A})$ . Applications to linear and nonlinear families of polynomials are briefly discussed.

## 2. BASIC NOTIONS OF ALGEBRAIC GEOMETRY

In this section we collect the basic definitions and facts of algebraic geometry that we need in the sequel. We use standard notions and notations which can be found in, e.g., [Kun85], [Sha94].

Let  $\mathbb{K}$  be any of the fields  $\mathbb{F}_q$  or  $\overline{\mathbb{F}}_q$ . We denote by  $\mathbb{A}^n$  the affine  $n$ -dimensional space  $\overline{\mathbb{F}}_q^n$  and by  $\mathbb{P}^n$  the projective  $n$ -dimensional space over  $\overline{\mathbb{F}}_q^{n+1}$ . Both spaces are endowed with their respective Zariski topologies over  $\mathbb{K}$ , for which a closed set is the zero locus of a set of polynomials of  $\mathbb{K}[X_1, \dots, X_n]$ , or of a set of homogeneous polynomials of  $\mathbb{K}[X_0, \dots, X_n]$ .

A subset  $V \subset \mathbb{P}^n$  is a *projective variety defined over  $\mathbb{K}$*  (or a projective  $\mathbb{K}$ -variety for short) if it is the set of common zeros in  $\mathbb{P}^n$  of homogeneous polynomials  $F_1, \dots, F_m \in \mathbb{K}[X_0, \dots, X_n]$ . Correspondingly, an *affine variety of  $\mathbb{A}^n$  defined over  $\mathbb{K}$*  (or an affine  $\mathbb{K}$ -variety) is the set of common zeros in  $\mathbb{A}^n$  of polynomials  $F_1, \dots, F_m \in \mathbb{K}[X_1, \dots, X_n]$ . We think a projective or affine  $\mathbb{K}$ -variety to be equipped with the induced Zariski topology. We shall denote by  $\{F_1 = 0, \dots, F_m = 0\}$  or  $V(F_1, \dots, F_m)$  the affine or projective  $\mathbb{K}$ -variety consisting of the common zeros of  $F_1, \dots, F_m$ .

In the remaining part of this section, unless otherwise stated, all results referring to varieties in general should be understood as valid for both projective and affine varieties.

A  $\mathbb{K}$ -variety  $V$  is *irreducible* if it cannot be expressed as a finite union of proper  $\mathbb{K}$ -subvarieties of  $V$ . Further,  $V$  is *absolutely irreducible* if it is  $\overline{\mathbb{F}}_q$ -irreducible as a  $\overline{\mathbb{F}}_q$ -variety. Any  $\mathbb{K}$ -variety  $V$  can be expressed as an irredundant union  $V = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_s$  of irreducible (absolutely irreducible)  $\mathbb{K}$ -varieties, unique up to reordering, which are called the *irreducible (absolutely irreducible)  $\mathbb{K}$ -components* of  $V$ .

For a  $\mathbb{K}$ -variety  $V$  contained in  $\mathbb{P}^n$  or  $\mathbb{A}^n$ , we denote by  $I(V)$  its *defining ideal*, namely the set of polynomials of  $\mathbb{K}[X_0, \dots, X_n]$ , or of  $\mathbb{K}[X_1, \dots, X_n]$ , vanishing on  $V$ . The *coordinate ring*  $\mathbb{K}[V]$  of  $V$  is defined as the quotient ring  $\mathbb{K}[X_0, \dots, X_n]/I(V)$  or  $\mathbb{K}[X_1, \dots, X_n]/I(V)$ . The *dimension*  $\dim V$  of a  $\mathbb{K}$ -variety  $V$  is the length  $r$  of a longest chain  $V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_r$  of nonempty irreducible  $\mathbb{K}$ -varieties contained in  $V$ . A  $\mathbb{K}$ -variety  $V$  is called *equidimensional* if all the irreducible  $\mathbb{K}$ -components of  $V$  are of the same dimension. In such a case, we say that  $V$  has *pure dimension*  $r$ , meaning that every irreducible  $\mathbb{K}$ -component of  $V$  has dimension  $r$ .

A  $\mathbb{K}$ -variety of  $\mathbb{P}^n$  or  $\mathbb{A}^n$  of pure dimension  $n - 1$  is called a  $\mathbb{K}$ -hypersurface. It turns out that a  $\mathbb{K}$ -hypersurface of  $\mathbb{P}^n$  (or  $\mathbb{A}^n$ ) is the set of zeros of a single nonzero polynomial of  $\mathbb{K}[X_0, \dots, X_n]$  (or of  $\mathbb{K}[X_1, \dots, X_n]$ ).

The *degree*  $\deg V$  of an irreducible  $\mathbb{K}$ -variety  $V$  is the maximum number of points lying in the intersection of  $V$  with a linear space  $L$  of codimension  $\dim V$ , for which  $V \cap L$  is a finite set. More generally, following [Hei83] (see also [Ful84]), if  $V = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_s$  is the decomposition of  $V$  into irreducible  $\mathbb{K}$ -components, we define the degree of  $V$  as

$$\deg V := \sum_{i=1}^s \deg \mathcal{C}_i.$$

The degree of a  $\mathbb{K}$ -hypersurface  $V$  is the degree of a polynomial of minimal degree defining  $V$ . Another property is that the degree of a dense open subset of a  $\mathbb{K}$ -variety  $V$  is equal to the degree of  $V$ .

An important tool for our estimates is the following *Bézout inequality* (see [Hei83], [Ful84], [Vog84]): if  $V$  and  $W$  are  $\mathbb{K}$ -varieties of the same ambient space, then the following inequality holds:

$$(2.1) \quad \deg(V \cap W) \leq \deg V \cdot \deg W.$$

Let  $V \subset \mathbb{A}^n$  be a  $\mathbb{K}$ -variety and  $I(V) \subset \mathbb{K}[X_1, \dots, X_n]$  its defining ideal. Let  $x$  be a point of  $V$ . The *dimension*  $\dim_x V$  of  $V$  at  $x$  is the maximum of the dimensions of the irreducible  $\mathbb{K}$ -components of  $V$  that contain  $x$ . If  $I(V) = (F_1, \dots, F_m)$ , the *tangent space*  $\mathcal{T}_x V$  to  $V$  at  $x$  is the kernel of the Jacobian matrix  $(\partial F_i / \partial X_j)_{1 \leq i \leq m, 1 \leq j \leq n}(x)$  of  $F_1, \dots, F_m$  with respect to  $X_1, \dots, X_n$  at  $x$ . We have the inequality  $\dim \mathcal{T}_x V \geq \dim_x V$  (see, e.g., [Sha94, page 94]). The point  $x$  is *regular* if  $\dim \mathcal{T}_x V = \dim_x V$ . Otherwise, the point  $x$  is called *singular*. The set of singular points of  $V$  is the *singular locus*  $\text{Sing}(V)$  of  $V$ ; it is a closed  $\mathbb{K}$ -subvariety of  $V$ . A variety is called *nonsingular* if its singular locus is empty. For a projective variety, the concepts of tangent space, regular and singular point can be defined by considering an affine neighborhood of the point under consideration.

Let  $V$  and  $W$  be irreducible affine  $\mathbb{K}$ -varieties of the same dimension and let  $f : V \rightarrow W$  be a regular map for which  $\overline{f(V)} = W$  holds, where  $\overline{f(V)}$  denotes the closure of  $f(V)$  with respect to the Zariski topology of  $W$ . Such a map is called *dominant*. Then  $f$  induces a ring extension  $\mathbb{K}[W] \hookrightarrow \mathbb{K}[V]$  by composition with  $f$ . We say that the dominant map  $f$  is *finite* if this extension is integral, namely each element  $\eta \in \mathbb{K}[V]$  satisfies a monic equation with coefficients in  $\mathbb{K}[W]$ . A basic fact is that a dominant finite morphism is necessarily closed. Another fact concerning dominant finite morphisms we shall use is that the preimage  $f^{-1}(S)$  of an irreducible closed subset  $S \subset W$  is of pure dimension  $\dim S$  (see, e.g., [Dan94, §4.2, Proposition]).

**2.1. Rational points.** Let  $\mathbb{P}^n(\mathbb{F}_q)$  be the  $n$ -dimensional projective space over  $\mathbb{F}_q$  and let  $\mathbb{A}^n(\mathbb{F}_q)$  be the  $n$ -dimensional  $\mathbb{F}_q$ -vector space  $\mathbb{F}_q^n$ . For a projective variety  $V \subset \mathbb{P}^n$  or an affine variety  $V \subset \mathbb{A}^n$ , we denote by  $V(\mathbb{F}_q)$  the set of  $\mathbb{F}_q$ -rational points of  $V$ , namely  $V(\mathbb{F}_q) := V \cap \mathbb{P}^n(\mathbb{F}_q)$  in the projective case and  $V(\mathbb{F}_q) := V \cap \mathbb{A}^n(\mathbb{F}_q)$  in the affine case. For an affine variety  $V$  of dimension  $r$  and degree  $\delta$ , we have (see, e.g., [CM06, Lemma 2.1])

$$(2.2) \quad |V(\mathbb{F}_q)| \leq \delta q^r.$$

On the other hand, if  $V$  is a projective variety of dimension  $r$  and degree  $\delta$ , we have (see [GL02, Proposition 12.1] or [CM07, Proposition 3.1]; see [LR15] for more precise upper bounds)

$$(2.3) \quad |V(\mathbb{F}_q)| \leq \delta p_r,$$

where  $p_r := q^r + q^{r-1} + \cdots + q + 1 = |\mathbb{P}^r(\mathbb{F}_q)|$ .

**2.2. Complete intersections.** Elements  $F_1, \dots, F_{n-r}$  in  $\mathbb{K}[X_1, \dots, X_n]$  or  $\mathbb{K}[X_0, \dots, X_n]$  form a *regular sequence* if  $F_1$  is nonzero and no  $F_i$  is zero or a zero divisor in the quotient ring  $\mathbb{K}[X_1, \dots, X_n]/(F_1, \dots, F_{i-1})$  or  $\mathbb{K}[X_0, \dots, X_n]/(F_1, \dots, F_{i-1})$  for  $2 \leq i \leq n-r$ . In that case, the (affine or projective)  $\mathbb{K}$ -variety  $V := V(F_1, \dots, F_{n-r})$  is called a *set-theoretic complete intersection*. We remark that  $V$  is necessarily of pure dimension  $r$ . Furthermore,  $V$  is called an *ideal-theoretic complete intersection* if its ideal  $I(V)$  over  $\mathbb{K}$  can be generated by  $n-r$  polynomials.

If  $V \subset \mathbb{P}^n$  is an ideal-theoretic complete intersection defined over  $\mathbb{K}$  of dimension  $r$ , and  $F_1, \dots, F_{n-r}$  is a system of homogeneous generators of  $I(V)$ , the degrees  $d_1, \dots, d_{n-r}$  depend only on  $V$  and not on the system of generators. Arranging the  $d_i$  in such a way that  $d_1 \geq d_2 \geq \cdots \geq d_{n-r}$ , we call  $(d_1, \dots, d_{n-r})$  the *multidegree* of  $V$ . In this case, a stronger version of (2.1) holds, called the *Bézout theorem* (see, e.g., [Har92, Theorem 18.3]):

$$\deg V = d_1 \cdots d_{n-r}.$$

In what follows we shall deal with a particular class of complete intersections, which we now define. A complete intersection  $V$  is called *normal* if it is *regular in codimension 1*, that is, the singular locus  $\text{Sing}(V)$  of  $V$  has codimension at least 2 in  $V$ , namely  $\dim V - \dim \text{Sing}(V) \geq 2$  (actually, normality is a general notion that agrees on complete intersections with the one we define here). A fundamental result for projective complete intersections is the Hartshorne connectedness theorem (see, e.g., [Kun85, Theorem VI.4.2]): if  $V \subset \mathbb{P}^n$  is a complete intersection defined over  $\mathbb{K}$  and  $W \subset V$  is any  $\mathbb{K}$ -subvariety of codimension at least 2, then  $V \setminus W$  is connected in the Zariski topology of  $\mathbb{P}^n$  over  $\mathbb{K}$ . Applying the Hartshorne connectedness theorem with  $W := \text{Sing}(V)$ , one deduces the following result.

**Theorem 2.1.** *If  $V \subset \mathbb{P}^n$  is a normal complete intersection, then  $V$  is absolutely irreducible.*

### 3. A GEOMETRIC APPROACH TO ESTIMATE VALUE SETS

Let  $m$  and  $d$  be positive integers with  $q > d \geq m+2$ , and let  $\mathcal{A}$  be the family of (1.3). We may assume without loss of generality that  $G_1, \dots, G_m$  are elements of  $\mathbb{F}_q[A_{d-1}, \dots, A_1]$ . Indeed, let  $\Pi : \mathcal{A} \rightarrow \mathbb{F}_q$  be the mapping  $\Pi(T^d + a_{d-1}T^{d-1} + \cdots + a_0) := a_0$ . Denote  $\mathcal{A}_{a_0} := \Pi^{-1}(a_0)$ . We have

$$\frac{1}{|\mathcal{A}|} \sum_{f \in \mathcal{A}} \mathcal{V}(f) - \mu_d q = \frac{1}{\sum_{a_0 \in \mathbb{F}_q} |\mathcal{A}_{a_0}|} \sum_{a_0 \in \mathbb{F}_q} |\mathcal{A}_{a_0}| \left( \frac{1}{|\mathcal{A}_{a_0}|} \sum_{f \in \mathcal{A}_{a_0}} \mathcal{V}(f) - \mu_d q \right).$$

As a consequence, if there exists a constant  $E(d_1, \dots, d_m, d)$  such that

$$\left| \frac{1}{|\mathcal{A}_{a_0}|} \sum_{f \in \mathcal{A}_{a_0}} \mathcal{V}(f) - \mu_d q \right| \leq E(d_1, \dots, d_m, d) q^{\frac{1}{2}}$$

holds for any  $a_0 \in \mathbb{F}_q$ , then we conclude that

$$\begin{aligned} \left| \frac{1}{|\mathcal{A}|} \sum_{f \in \mathcal{A}} \mathcal{V}(f) - \mu_d q \right| &\leq \frac{1}{\sum_{a_0 \in \mathbb{F}_q} |\mathcal{A}_{a_0}|} \sum_{a_0 \in \mathbb{F}_q} |\mathcal{A}_{a_0}| E(d_1, \dots, d_m, d) q^{\frac{1}{2}} \\ &\leq E(d_1, \dots, d_m, d) q^{\frac{1}{2}}. \end{aligned}$$

Further, as  $\mathcal{V}(f) = \mathcal{V}(f + a_0)$  for any  $f \in \mathcal{A}$ , we shall also assume that  $f(0) = 0$  for any  $f \in \mathcal{A}$ .

Observe that, given  $f \in \mathcal{A}$ ,  $\mathcal{V}(f)$  equals the number of  $a_0 \in \mathbb{F}_q$  for which  $f + a_0$  has at least one root in  $\mathbb{F}_q$ . If  $\mathbb{K}$  is any of the fields  $\mathbb{F}_q$  or  $\overline{\mathbb{F}_q}$ , by  $\mathbb{K}[T]_d$  we denote the set of monic polynomials of  $\mathbb{K}[T]$  of degree  $d$ . Let  $\mathcal{N} : \mathbb{F}_q[T]_d \rightarrow \mathbb{Z}_{\geq 0}$  be the counting function of the number of roots in  $\mathbb{F}_q$  and  $\mathbf{1}_{\{\mathcal{N} > 0\}} : \mathbb{F}_q[T]_d \rightarrow \{0, 1\}$  the characteristic function of the set of polynomials having at least one root in  $\mathbb{F}_q$ . We deduce that

$$\begin{aligned} \sum_{f \in \mathcal{A}} \mathcal{V}(f) &= \sum_{a_0 \in \mathbb{F}_q} \sum_{f \in \mathcal{A}} \mathbf{1}_{\{\mathcal{N} > 0\}}(f + a_0) \\ &= |\{f + a_0 \in \mathcal{A} + \mathbb{F}_q : \mathcal{N}(f + a_0) > 0\}|. \end{aligned}$$

For a set  $\mathcal{X} \subset \mathbb{F}_q$ , we define  $\mathcal{S}_{\mathcal{X}}^{\mathcal{A}} \subset \mathbb{F}_q[T]_d$  as the set of polynomials  $f + a_0 \in \mathcal{A} + \mathbb{F}_q$  vanishing on  $\mathcal{X}$ , namely

$$\mathcal{S}_{\mathcal{X}}^{\mathcal{A}} := \{f + a_0 \in \mathcal{A} + \mathbb{F}_q : (f + a_0)(x) = 0 \text{ for any } x \in \mathcal{X}\}.$$

For  $r \in \mathbb{N}$  we shall use the symbol  $\mathcal{X}_r$  to denote a subset of  $\mathbb{F}_q$  of  $r$  elements. Our approach to determine the asymptotic behavior of  $\mathcal{V}(\mathcal{A})$  relies on the following combinatorial result.

**Lemma 3.1.** *Given  $d, m \in \mathbb{N}$  with  $q > d \geq m + 2$ , we have*

$$(3.1) \quad \mathcal{V}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}|.$$

*Proof.* Given a subset  $\mathcal{X}_r := \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{F}_q$ , consider the set  $\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}} \subset \mathbb{F}_q[T]_d$  defined as above. It is easy to see that  $\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}} = \bigcap_{i=1}^r \mathcal{S}_{\{\alpha_i\}}^{\mathcal{A}}$  and

$$|\{f + a_0 \in \mathcal{A} + \mathbb{F}_q : \mathcal{N}(f + a_0) > 0\}| = \left| \bigcup_{x \in \mathbb{F}_q} \mathcal{S}_{\{x\}}^{\mathcal{A}} \right|.$$

Therefore the inclusion-exclusion principle implies

$$\mathcal{V}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \left| \bigcup_{x \in \mathbb{F}_q} \mathcal{S}_{\{x\}}^{\mathcal{A}} \right| = \frac{1}{|\mathcal{A}|} \sum_{r=1}^q (-1)^{r-1} \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}|.$$

Now  $|\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}| = 0$  for  $r > d$ , because a polynomial of degree  $d$  cannot vanish on more than  $d$  elements of  $\mathbb{F}_q$ . This readily implies the lemma.  $\square$

Lemma 3.1 shows that the behavior of  $\mathcal{V}(\mathcal{A})$  is determined by that of

$$(3.2) \quad \mathcal{S}_r^{\mathcal{A}} := \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}|,$$

for  $1 \leq r \leq d$ , which are the subject of the next sections.

**3.1. A geometric approach to estimate  $\mathcal{S}_r^{\mathcal{A}}$ .** Fix  $r$  with  $1 \leq r \leq d$ . Let  $A_{d-1}, \dots, A_0$  be indeterminates over  $\overline{\mathbb{F}}_q$  and let  $G_1, \dots, G_m \in \mathbb{F}_q[A_{d-1}, \dots, A_1]$  be the polynomials defining the family  $\mathcal{A}$  of (1.3). Set  $\mathbf{A} := (A_{d-1}, \dots, A_1)$  and  $\mathbf{A}_0 := (\mathbf{A}, A_0)$ . To estimate  $\mathcal{S}_r^{\mathcal{A}}$ , we introduce the following definitions and notations. Let  $T, T_1, \dots, T_r$  be new indeterminates over  $\overline{\mathbb{F}}_q$  and denote  $\mathbf{T} := (T_1, \dots, T_r)$ . Consider the polynomial  $F \in \mathbb{F}_q[\mathbf{A}_0, T]$  defined as

$$(3.3) \quad F(\mathbf{A}_0, T) := T^d + A_{d-1}T^{d-1} + \dots + A_1T + A_0.$$

Observe that if  $\mathbf{a}_0 \in \mathbb{F}_q^d$ , then we may write  $F(\mathbf{a}_0, T) = f + a_0$ , where  $f \in \mathbb{F}_q[T]_d$  and  $f(0) = 0$ .

Consider the affine quasi- $\mathbb{F}_q$ -variety  $\Gamma_r \subset \mathbb{A}^{d+r}$  defined as follows:

$$\begin{aligned} \Gamma_r := \{(\mathbf{a}, a_0, \boldsymbol{\alpha}) \in \mathbb{A}^d \times \mathbb{A}^r : & \alpha_i \neq \alpha_j \ (1 \leq i < j \leq r), \\ & F(\mathbf{a}_0, \alpha_i) = 0 \ (1 \leq i \leq r), \ G_k(\mathbf{a}) = 0 \ (1 \leq k \leq m)\}. \end{aligned}$$

Our next result explains how the number  $|\Gamma_r(\mathbb{F}_q)|$  of  $\mathbb{F}_q$ -rational points of  $\Gamma_r$  is related to the numbers  $\mathcal{S}_r^{\mathcal{A}}$  ( $1 \leq r \leq d$ ).

**Lemma 3.2.** *Let  $r$  be an integer with  $1 \leq r \leq d$ . Then*

$$\frac{|\Gamma_r(\mathbb{F}_q)|}{r!} = \mathcal{S}_r^{\mathcal{A}}.$$

*Proof.* Let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be a point of  $\Gamma_r(\mathbb{F}_q)$  and  $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, r\}$  an arbitrary permutation. Let  $\sigma(\boldsymbol{\alpha})$  be the image of  $\boldsymbol{\alpha}$  by the linear mapping induced by  $\sigma$ . Then it is clear that  $(\mathbf{a}_0, \sigma(\boldsymbol{\alpha}))$  belong to  $\Gamma_r(\mathbb{F}_q)$ . Furthermore,  $\sigma(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$  if and only if  $\sigma$  is the identity permutation. This shows that  $\mathbb{S}_r$ , the symmetric group of  $r$  elements, acts over the set  $\Gamma_r(\mathbb{F}_q)$  and each orbit under this action has  $r!$  elements.

The orbit of an arbitrary point  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r(\mathbb{F}_q)$  uniquely determines a polynomial  $F(\mathbf{a}_0, T) = f + a_0$  with  $f \in \mathcal{A}$  and a set  $\mathcal{X}_r := \{\alpha_1, \dots, \alpha_r\} \subset \mathbb{F}_q$  with  $|\mathcal{X}_r| = r$  such that  $(f + a_0)|_{\mathcal{X}_r} \equiv 0$ . Therefore, each orbit uniquely determines a set  $\mathcal{X}_r \subset \mathbb{F}_q$  with  $|\mathcal{X}_r| = r$  and an element of  $\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}$ . Reciprocally, to each element of  $\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}$  corresponds a unique orbit of  $\Gamma_r(\mathbb{F}_q)$ . This implies

$$\text{number of orbits of } \Gamma_r(\mathbb{F}_q) = \sum_{\mathcal{X}_r \subseteq \mathbb{F}_q} |\mathcal{S}_{\mathcal{X}_r}^{\mathcal{A}}|,$$

and finishes the proof of the lemma.  $\square$

To estimate the quantity  $|\Gamma_r(\mathbb{F}_q)|$  we shall consider the Zariski closure  $\text{cl}(\Gamma_r)$  of  $\Gamma_r \subset \mathbb{A}^{d+r}$ . Our aim is to provide explicit equations defining  $\text{cl}(\Gamma_r)$ . For this purpose, we shall use the following notation. Let  $X_1, \dots, X_{l+1}$  be indeterminates over  $\overline{\mathbb{F}}_q$  and  $f \in \overline{\mathbb{F}}_q[T]$  a polynomial of degree at most  $l$ . For notational convenience, we define the 0th divided difference  $\Delta^0 f \in \overline{\mathbb{F}}_q[X_1]$  of  $f$  as  $\Delta^0 f := f(X_1)$ . Further, for  $1 \leq i \leq l$  we define the  $i$ th divided difference  $\Delta^i f \in \overline{\mathbb{F}}_q[X_1, \dots, X_{i+1}]$  of  $f$  as

$$\Delta^i f(X_1, \dots, X_{i+1}) = \frac{\Delta^{i-1} f(X_1, \dots, X_i) - \Delta^{i-1} f(X_1, \dots, X_{i-1}, X_{i+1})}{X_i - X_{i+1}}.$$

With these notations, let  $\Gamma_r^* \subset \mathbb{A}^{d+r}$  be the  $\mathbb{F}_q$ -variety defined as

$$\begin{aligned} \Gamma_r^* := \{(\mathbf{a}_0, \boldsymbol{\alpha}) \in \mathbb{A}^d \times \mathbb{A}^r : & \Delta^{i-1} F(\mathbf{a}_0, \alpha_1, \dots, \alpha_i) = 0 \ (1 \leq i \leq r), \\ & G_k(\mathbf{a}_0) = 0 \ (1 \leq k \leq m)\}, \end{aligned}$$

where  $\Delta^{i-1} F(\mathbf{a}_0, T_1, \dots, T_i)$  denotes the  $(i-1)$ -divided difference of  $F(\mathbf{a}_0, T) \in \overline{\mathbb{F}}_q[T]$ . Next we relate the varieties  $\Gamma_r$  and  $\Gamma_r^*$ .

**Lemma 3.3.** *With notations and assumptions as above, we have*

$$(3.4) \quad \Gamma_r = \Gamma_r^* \cap \{(\mathbf{a}_0, \boldsymbol{\alpha}) : \alpha_i \neq \alpha_j \ (1 \leq i < j \leq r)\}.$$



*Proof.* Let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be a point of  $\Gamma_r$ . By the definition of the divided differences of  $F(\mathbf{a}_0, T)$  we easily conclude that  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$ . On the other hand, let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be a point belonging to the set in the right-hand side of (3.4). We claim that  $F(\mathbf{a}_0, \alpha_i) = 0$  for  $1 \leq i \leq r$ . We observe that  $F(\mathbf{a}_0, \alpha_1) = \Delta^0 F(\mathbf{a}_0, \alpha_1) = 0$ . Arguing inductively, suppose that we have  $F(\mathbf{a}_0, \alpha_1) = \dots = F(\mathbf{a}_0, \alpha_{i-1}) = 0$ . By definition  $\Delta^{i-1} F(\mathbf{a}_0, \alpha_1, \dots, \alpha_i)$  can be expressed as a linear combination with nonzero coefficients of the differences  $F(\mathbf{a}_0, \alpha_{j+1}) - F(\mathbf{a}_0, \alpha_j)$  with  $1 \leq j \leq i-1$ . Combining the inductive hypothesis with the fact that  $\Delta^{i-1} F(\mathbf{a}_0, \alpha_1, \dots, \alpha_i) = 0$ , we easily conclude that  $F(\mathbf{a}_0, \alpha_i) = 0$ , finishing thus the proof of the claim.  $\square$

#### 4. ON THE GEOMETRY OF THE VARIETY $\Gamma_r^*$

In this section we establish several properties of the geometry of the affine  $\mathbb{F}_q$ -variety  $\Gamma_r^*$ , assuming that the polynomials  $G_1, \dots, G_m$  and the affine variety  $V \subset \mathbb{A}^d$  they define satisfy certain conditions that we now state. The first conditions allow us to estimate the cardinality of  $\mathcal{A}$ :

- (H<sub>1</sub>)  $G_1, \dots, G_m$  form a regular sequence and generate a radical ideal of  $\mathbb{F}_q[A_{d-1}, \dots, A_0]$ .
- (H<sub>2</sub>) The variety  $V \subset \mathbb{A}^d$  defined by  $G_1, \dots, G_m$  is normal.
- (H<sub>3</sub>) Let  $G_1^{d_1}, \dots, G_m^{d_m}$  denote the homogeneous parts of higher degree of  $G_1, \dots, G_m$ . Then  $G_1^{d_1}, \dots, G_m^{d_m}$  satisfy (H<sub>1</sub>) and (H<sub>2</sub>).

As stated in the introduction, we are interested in families  $\mathcal{A}$  for which  $\mathcal{V}(\mathcal{A}) = \mu_d q + \mathcal{O}(q^{\frac{1}{2}})$ . If many of the polynomials in  $\mathcal{A} + \mathbb{F}_q$  are not square-free, then  $\mathcal{V}(\mathcal{A})$  might not behave as in the general case. For  $\mathcal{B} \subset \overline{\mathbb{F}_q}[T]_d$ , the set of elements of  $\mathcal{B}$  which are not square-free is called the *discriminant locus*  $\mathcal{D}(\mathcal{B})$  of  $\mathcal{B}$ . With a slight abuse of notation, in what follows we identify each  $f_{\mathbf{a}_0} = T^d + a_{d-1}T^{d-1} + \dots + a_0 \in \mathcal{B}$  with the  $d$ -tuple  $\mathbf{a}_0 := (a_{d-1}, \dots, a_0)$ , and consider  $\mathcal{B}$  as a subset of  $\mathbb{A}^d$ . For  $f_{\mathbf{a}_0} \in \mathcal{B}$ , let  $\text{Disc}(f_{\mathbf{a}_0}) := \text{Res}(f_{\mathbf{a}_0}, f'_{\mathbf{a}_0})$  denote the discriminant of  $f_{\mathbf{a}_0}$ , that is, the resultant of  $f_{\mathbf{a}_0}$  and its derivative  $f'_{\mathbf{a}_0}$ . Since  $f_{\mathbf{a}_0}$  has degree  $d$ , by basic properties of resultants it follows that

$$\begin{aligned} \text{Disc}(f_{\mathbf{a}_0}) &= \text{Disc}(F(\mathbf{A}_0, T))|_{\mathbf{A}_0=\mathbf{a}_0} \\ &:= \text{Res}(F(\mathbf{A}_0, T), \Delta^1 F(\mathbf{A}_0, T, T), T)|_{\mathbf{A}_0=\mathbf{a}_0}, \end{aligned}$$

where the expression  $\text{Res}$  in the right-hand side denotes resultant with respect to  $T$ . Observe that  $\mathcal{D}(\mathcal{B}) = \{\mathbf{a}_0 \in \mathcal{B} : \text{Disc}(F(\mathbf{A}_0, T))|_{\mathbf{A}_0=\mathbf{a}_0} = 0\}$ .

We shall need further to consider first subdiscriminant loci. The *first subdiscriminant locus*  $\mathcal{S}_1(\mathcal{B})$  of  $\mathcal{B} \subset \overline{\mathbb{F}_q}[T]_d$  is the set of  $\mathbf{a}_0 \in \mathcal{B}$  for which the first subdiscriminant  $\text{Subdisc}(f_{\mathbf{a}_0}) := \text{Subres}(f_{\mathbf{a}_0}, f'_{\mathbf{a}_0})$  vanishes, where  $\text{Subres}(f_{\mathbf{a}_0}, f'_{\mathbf{a}_0})$  denotes the first subresultant of  $f_{\mathbf{a}_0}$  and  $f'_{\mathbf{a}_0}$ . Since  $f_{\mathbf{a}_0}$  has degree  $d$ , basic properties of subresultants imply

$$\begin{aligned} \text{Subdisc}(f_{\mathbf{a}_0}) &= \text{Subdisc}(F(\mathbf{A}_0, T))|_{\mathbf{A}_0=\mathbf{a}_0} \\ &:= \text{Subres}(F(\mathbf{A}_0, T), \Delta^1 F(\mathbf{A}_0, T, T), T)|_{\mathbf{A}_0=\mathbf{a}_0}, \end{aligned}$$

where  $\text{Subres}$  in the right-hand side denotes first subresultant with respect to  $T$ . We have  $\mathcal{S}_1(\mathcal{B}) = \{\mathbf{a}_0 \in \mathcal{B} : \text{Subdisc}(F(\mathbf{A}_0, T))|_{\mathbf{A}_0=\mathbf{a}_0} = 0\}$ .

Our next condition requires that the discriminant and the first subdiscriminant locus intersect well  $V$ . More precisely, we require the condition:

- (H<sub>4</sub>)  $V \cap \mathcal{D}(V)$  has codimension one in  $V$ , and  $V \cap \mathcal{D}(V) \cap \mathcal{S}_1(V)$  has codimension two in  $V$ .

We shall prove that  $\Gamma_r^*$  is a set-theoretic complete intersection, whose singular locus has codimension at least 2. This will allow us to conclude that  $\Gamma_r^*$  is an ideal-theoretic complete intersection.

**Lemma 4.1.**  $\Gamma_r^*$  is a set-theoretic complete intersection of dimension  $d - m$ .

*Proof.* By hypothesis  $(H_1)$ ,  $G_1, \dots, G_m$  form a regular sequence. In order to prove that  $G_1, \dots, G_m, \Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ) form a regular sequence, we argue by induction on  $i$ .

For  $i = 1$ , we observe that the set of common zeros of  $G_1, \dots, G_m$  in  $\mathbb{A}^d \times \mathbb{A}^r$  is  $V \times \mathbb{A}^r$ , and each irreducible component of  $V \times \mathbb{A}^r$  is of the form  $\mathcal{C} \times \mathbb{A}^r$ , where  $\mathcal{C}$  is an irreducible component of  $V$ . As  $\Delta^0 F(\mathbf{A}_0, T_1) = F(\mathbf{A}_0, T_1)$  is of degree  $d$  in  $T_1$ , it cannot vanish identically on any component  $\mathcal{C} \times \mathbb{A}^r$ , which implies that it cannot be a zero divisor modulo  $G_1, \dots, G_m$ .

Now suppose that the assertion is proved for  $1 \leq j \leq r - 1$ , that is, the polynomials  $G_1, \dots, G_m, \Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq j$ ) form a regular sequence. These are all elements of  $\mathbb{F}_q[\mathbf{A}_0, T_1, \dots, T_j]$ . On the other hand, the monomial  $T_{j+1}^{d-j}$  occurs in the dense representation of  $\Delta^j F(\mathbf{A}_0, T_1, \dots, T_{j+1})$  with a nonzero coefficient. We deduce that  $\Delta^j F(\mathbf{A}_0, T_1, \dots, T_{j+1})$  cannot be a zero divisor modulo  $G_1, \dots, G_m, \Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq j$ ), which finishes the proof of our assertion. This implies the statement of the lemma.  $\square$

**4.1. The dimension of the singular locus of  $\Gamma_r^*$ .** Next we show that the singular locus of  $\Gamma_r^*$  has codimension at least 2 in  $\Gamma_r^*$ . For this purpose, we carry out an analysis of the singular locus of  $\Gamma_r^*$  which generalizes that of [MPP14, Section 4.1] to this (eventually nonlinear) setting. We start with the following criterion of nonsingularity.

**Lemma 4.2.** *Let  $J_{\mathbf{G}, \mathbf{F}} \in \overline{\mathbb{F}}_q[\mathbf{A}_0, \mathbf{T}]^{(m+r) \times (d+r)}$  be the Jacobian matrix of  $\mathbf{G} := (G_1, \dots, G_m)$  and  $F(\mathbf{A}_0, T_i)$  ( $1 \leq i \leq r$ ) with respect to  $\mathbf{A}_0, \mathbf{T}$ , and let  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$ . If  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has full rank, then  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is a nonsingular point of  $\Gamma_r^*$ .*

*Proof.* Considering the Newton form of the polynomial interpolating  $F(\mathbf{a}_0, T)$  at  $\alpha_1, \dots, \alpha_r$  we easily deduce that  $F(\mathbf{a}_0, \alpha_i) = 0$  for  $1 \leq i \leq r$ . This shows that  $F(\mathbf{A}_0, T_i)$  vanishes on  $\Gamma_r^*$  for  $1 \leq i \leq r$ . As a consequence, any element of the tangent space  $\mathcal{T}_{(\mathbf{a}_0, \boldsymbol{\alpha})} \Gamma_r^*$  of  $\Gamma_r^*$  at  $(\mathbf{a}_0, \boldsymbol{\alpha})$  belongs to the kernel of the Jacobian matrix  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$ .

By hypothesis, the  $(m+r) \times (d+r)$  matrix  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has full rank  $m+r$ , and thus its kernel has dimension  $d-m$ . We conclude that the tangent space  $\mathcal{T}_{(\mathbf{a}_0, \boldsymbol{\alpha})} \Gamma_r^*$  has dimension at most  $d-m$ . Since  $\Gamma_r^*$  is of pure dimension  $d-m$ , it follows that  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is a nonsingular point of  $\Gamma_r^*$ .  $\square$

Let  $(\mathbf{a}_0, \boldsymbol{\alpha}) := (\mathbf{a}_0, \alpha_1, \dots, \alpha_r)$  be an arbitrary point of  $\Gamma_r^*$  and let  $f_{\mathbf{a}_0} := F(\mathbf{a}_0, T)$ . Then the Jacobian matrix  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has the following form:

$$J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha}) := \begin{pmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{A}_0}(\mathbf{a}_0, \boldsymbol{\alpha}) & \mathbf{0} \\ \frac{\partial F}{\partial \mathbf{A}_0}(\mathbf{a}_0, \boldsymbol{\alpha}) & \frac{\partial F}{\partial \mathbf{T}}(\mathbf{a}_0, \boldsymbol{\alpha}) \end{pmatrix}.$$

Observe that  $(\partial F / \partial \mathbf{T})(\mathbf{a}_0, \boldsymbol{\alpha})$  is a diagonal matrix whose  $i$ th diagonal entry is  $f'_{\mathbf{a}_0}(\alpha_i)$ . If  $(\partial \mathbf{G} / \partial \mathbf{A}_0)(\mathbf{a}_0, \boldsymbol{\alpha})$  has full rank and all the roots in  $\overline{\mathbb{F}}_q$  of  $f_{\mathbf{a}_0}$  are simple, then  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has full rank and  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is a regular point of  $\Gamma_r^*$ . Therefore, to prove that the singular locus of  $\Gamma_r^*$  is a subvariety of codimension at least 2 in  $\Gamma_r^*$ , it suffices to consider the set of points  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  for which either  $\mathbf{a}_0$  is a singular point of  $V$ , or at least one coordinate of  $\boldsymbol{\alpha}$  is a multiple root of  $f_{\mathbf{a}_0}$ . To deal with the first of these cases, consider the morphism of  $\mathbb{F}_q$ -varieties defined as follows:

$$(4.1) \quad \begin{array}{ccc} \Psi_r : & \Gamma_r^* & \rightarrow V \\ & (\mathbf{a}_0, \boldsymbol{\alpha}) & \mapsto \mathbf{a}_0, \end{array}$$

We have the following result.

**Lemma 4.3.**  *$\Psi_r$  is a finite dominant morphism.*



*Proof.* It is easy to see that  $\Psi_r$  is a surjective mapping. Therefore, it suffices to show that the coordinate function  $t_i$  of  $\overline{\mathbb{F}}_q[\Gamma_r^*]$  defined by  $T_i$  satisfies a monic equation with coefficients in  $\overline{\mathbb{F}}_q[V]$  for  $1 \leq i \leq r$ . Denote by  $\xi_j$  the coordinate function of  $V$  defined by  $A_j$  for  $0 \leq j \leq d-1$ , and set  $\xi_0 := (\xi_{d-1}, \dots, \xi_0)$ . Since the polynomial  $F(\mathbf{A}_0, T_i)$  vanishes on  $\Gamma_r^*$  for  $1 \leq i \leq r$ , and is a monic element of  $\overline{\mathbb{F}}_q[\mathbf{A}_0][T_i]$ , we deduce that the monic element  $F(\xi_0, T_i)$  of  $\overline{\mathbb{F}}_q[V][T_i]$  annihilates  $t_i$  in  $\Gamma_r^*$  for  $1 \leq i \leq r$ . This shows that the ring extension  $\overline{\mathbb{F}}_q[V] \hookrightarrow \overline{\mathbb{F}}_q[\Gamma_r^*]$  is integral, namely  $\Psi_r$  is a finite dominant mapping.  $\square$

A first consequence of Lemma 4.3 is that the set of points  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  with  $\mathbf{a}_0$  singular are under control.

**Corollary 4.4.** *The set  $\mathcal{W}_0$  of points  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  with  $\mathbf{a}_0 \in \text{Sing}(V)$  is contained in a subvariety of codimension 2 of  $\Gamma_r^*$ .*

*Proof.* Hypotheses  $(H_1)$  and  $(H_2)$  imply that  $V$  is a normal complete intersection. It follows that  $\text{Sing}(V)$  has codimension at least two in  $V$ . Then  $\mathcal{W}_0 = \Psi_r^{-1}(\text{Sing}(V))$  has codimension at least two in  $\Gamma_r^*$ .  $\square$

By Corollary 4.4 it suffices to consider the set of singular points  $(\mathbf{a}_0, \boldsymbol{\alpha})$  of  $\Gamma_r^*$  with  $\mathbf{a}_0 \in V \setminus \text{Sing}(V)$ . By the remarks before Lemma 4.3, if  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is such a singular point, then  $f_{\mathbf{a}_0}$  must have multiple roots. We start considering the “extreme” case where  $f'_{\mathbf{a}_0}$  is the zero polynomial.

**Lemma 4.5.** *The set  $\mathcal{W}_1$  of points  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  with  $f'_{\mathbf{a}_0} = 0$  is contained in a subvariety of codimension 2 of  $\Gamma_r^*$ .*

*Proof.* The condition  $f'_{\mathbf{a}_0} = 0$  implies  $\mathbf{a}_0$  belongs to both the discriminant locus  $\mathcal{D}(V)$  and the first subdiscriminant locus  $\mathcal{S}_1(V)$ , namely  $\mathcal{W}_1 \subset \Psi_r^{-1}(\mathcal{D}(V) \cap \mathcal{S}_1(V))$ . Hypothesis  $(H_4)$  asserts that  $\mathcal{D}(V) \cap \mathcal{S}_1(V)$  has codimension two in  $V$ . Therefore, taking into account that  $\Psi_r$  is a finite morphism we deduce that  $\mathcal{W}_1$  has codimension two in  $\Gamma_r^*$ .  $\square$

In what follows we shall assume that  $\mathbf{a}_0$  is a regular point of  $V$ ,  $f'_{\mathbf{a}_0}$  is nonzero and  $f_{\mathbf{a}_0}$  has multiple roots. We analyze the case where exactly one of the coordinates of  $\boldsymbol{\alpha}$  is a multiple root of  $f_{\mathbf{a}_0}$ .

**Lemma 4.6.** *Suppose that there exists a unique coordinate  $\alpha_i$  of  $\boldsymbol{\alpha}$  which is a multiple root of  $f_{\mathbf{a}_0}$ . Then  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is a regular point of  $\Gamma_r^*$ .*

*Proof.* Assume without loss of generality that  $\alpha_1$  is the only multiple root of  $f_{\mathbf{a}_0}$  among the coordinates of  $\boldsymbol{\alpha}$ . According to Lemma 4.2, it suffices to show that the Jacobian matrix  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has full rank. For this purpose, consider the  $r \times (r+1)$ -submatrix  $\partial F / \partial (A_0, \mathbf{T})(\mathbf{a}_0, \boldsymbol{\alpha})$  of  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  consisting of the entries of its last  $r$  rows and its last  $r+1$  columns:

$$\frac{\partial F}{\partial (A_0, \mathbf{T})}(\mathbf{a}_0, \boldsymbol{\alpha}) := \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & f'_{\mathbf{a}_0}(\alpha_2) & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & 0 & \cdots & 0 & f'_{\mathbf{a}_0}(\alpha_r) \end{pmatrix}.$$

Since by hypothesis  $\alpha_i$  is a simple root of  $f'_{\mathbf{a}_0}$  for  $i \geq 2$ , we have  $f'_{\mathbf{a}_0}(\alpha_i) \neq 0$  for  $i \geq 2$ , and thus  $\partial F / \partial (A_0, \mathbf{T})(\mathbf{a}_0, \boldsymbol{\alpha})$  is of full rank  $r$ .

On the other hand, since the matrix  $(\partial \mathbf{G} / \partial \mathbf{A}_0)(\mathbf{a}_0, \boldsymbol{\alpha})$  has rank  $m$  and its last column is zero, denoting by  $(\partial \mathbf{G} / \partial \mathbf{A})(\mathbf{a}_0, \boldsymbol{\alpha})$  the submatrix of  $(\partial \mathbf{G} / \partial \mathbf{A}_0)(\mathbf{a}_0, \boldsymbol{\alpha})$  obtained by deleting

its last column, we see that  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  can be expressed as the following block matrix:

$$J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha}) = \begin{pmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{A}}(\mathbf{a}_0, \boldsymbol{\alpha}) & \mathbf{0} \\ * & \frac{\partial F}{\partial(A_0, \mathbf{T})}(\mathbf{a}_0, \boldsymbol{\alpha}) \end{pmatrix}.$$

Since both  $(\partial \mathbf{G} / \partial \mathbf{A})(\mathbf{a}_0, \boldsymbol{\alpha})$  and  $(\partial F / \partial(A_0, \mathbf{T}))(\mathbf{a}_0, \boldsymbol{\alpha})$  have full rank, we conclude that  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \boldsymbol{\alpha})$  has rank  $m + r$ .  $\square$

Now we analyze the case where two distinct multiple roots of  $f_{\mathbf{a}_0}$  occur among the coordinates of  $\boldsymbol{\alpha}$ .

**Lemma 4.7.** *Let  $\mathcal{W}_2$  be the set of points  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  for which there exist  $1 \leq i < j \leq r$  such that  $\alpha_i \neq \alpha_j$  and  $\alpha_i, \alpha_j$  are multiple roots of  $f_{\mathbf{a}_0}$ . Then  $\mathcal{W}_2$  is contained in a subvariety of codimension 2 of  $\Gamma_r^*$ .*

*Proof.* Let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be an arbitrary point of  $\mathcal{W}_2$ . Since  $f_{\mathbf{a}_0}$  has at least two distinct multiple roots, the greatest common divisor of  $f_{\mathbf{a}_0}$  and  $f'_{\mathbf{a}_0}$  has degree at least 2. This implies that

$$\text{Disc}(f_{\mathbf{a}_0}) = \text{Subdisc}(f_{\mathbf{a}_0}) = 0.$$

As a consequence,  $\mathcal{W}_2 \subset \Psi_r^{-1}(\mathcal{Z})$ , where  $\Psi_r$  is the morphism of (4.1),  $\mathcal{Z} := \mathcal{D}(V) \cap \mathcal{S}_1(V)$  and  $\mathcal{D}(V)$  and  $\mathcal{S}_1(V)$  are the discriminant locus and the first subdiscriminant locus of  $V$ . Hypothesis  $(H_4)$  proves that  $\mathcal{Z}$  has codimension two in  $V$ . It follows that  $\dim \mathcal{Z} = d - m - 2$ , and hence  $\dim \Psi_r^{-1}(\mathcal{Z}) = d - m - 2$ . The statement of the lemma follows.  $\square$

Next we consider the case where only one multiple root of  $f_{\mathbf{a}_0}$  occurs among the coordinates of  $\boldsymbol{\alpha}$ , and at least two distinct coordinates of  $\boldsymbol{\alpha}$  take this value. Then we have either that all the remaining coordinates of  $\boldsymbol{\alpha}$  are simple roots of  $f_{\mathbf{a}_0}$ , or there exists at least a third coordinate whose value is the same multiple root. We now deal with the first of these two cases.

**Lemma 4.8.** *Let  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  be a point satisfying the conditions:*

- *there exist  $1 \leq i < j \leq r$  such that  $\alpha_i = \alpha_j$  and  $\alpha_i$  is a multiple root of  $f_{\mathbf{a}_0}$ ;*
- *for any  $k \notin \{i, j\}$ ,  $\alpha_k$  is a simple root of  $f_{\mathbf{a}_0}$ .*

*Then either  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is regular point of  $\Gamma_r^*$  or it is contained in a subvariety  $\mathcal{W}_3$  of codimension two of  $\Gamma_r^*$ .*

*Proof.* We may assume without loss of generality that  $i = 1$  and  $j = 2$ . Observe that  $\Delta^1 F(\mathbf{A}_0, T_1, T_2)$  and  $F(\mathbf{A}_0, T_i)$  ( $2 \leq i \leq r$ ) vanish on  $\Gamma_r^*$ . Therefore, the tangent space  $\mathcal{T}_{(\mathbf{a}_0, \boldsymbol{\alpha})} \Gamma_r^*$  of  $\Gamma_r^*$  at  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is included in the kernel of the Jacobian matrix  $J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha})$  of  $\mathbf{G}$ ,  $\Delta^1 F(\mathbf{A}_0, T_1, T_2)$  and  $F(\mathbf{A}_0, T_i)$  ( $2 \leq i \leq r$ ) with respect to  $\mathbf{A}_0, \mathbf{T}$ . We claim that either  $J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha})$  has rank  $r + m$ , or  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is contained in a subvariety of codimension two of  $\Gamma_r^*$ .

Now we prove the claim. We may express  $J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha})$  as

$$J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha}) = \begin{pmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{A}}(\mathbf{a}_0, \boldsymbol{\alpha}) & \mathbf{0} \\ * & \frac{\partial(\Delta^1, \mathbf{F}^*)}{\partial(A_0, \mathbf{T})}(\mathbf{a}_0, \boldsymbol{\alpha}) \end{pmatrix},$$

where  $(\partial \mathbf{G} / \partial \mathbf{A})(\mathbf{a}_0, \boldsymbol{\alpha}) \in \mathbb{F}_q^{m \times (d-1)}$  is the Jacobian matrix of  $\mathbf{G}$  with respect to  $\mathbf{A}$  and  $(\partial(\Delta^1, \mathbf{F}^*) / \partial(A_0, \mathbf{T}))(\mathbf{a}_0, \boldsymbol{\alpha}) \in \mathbb{F}_q^{r \times (r+1)}$  is the Jacobian matrix of  $\Delta^1 F(\mathbf{A}_0, T_1, T_2)$  and

$F(\mathbf{A}_0, T_i)$  ( $2 \leq i \leq r$ ) with respect to  $\mathbf{A}_0, \mathbf{T}$ :

$$\frac{\partial(\Delta^1, \mathbf{F}^*)}{\partial(\mathbf{A}_0, \mathbf{T})}(\mathbf{a}_0, \boldsymbol{\alpha}) := \begin{pmatrix} 0 & \lambda_1 & \lambda_2 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & f'_{\mathbf{a}_0}(\alpha_3) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 1 & 0 & 0 & 0 & \cdots & f'_{\mathbf{a}_0}(\alpha_r) \end{pmatrix}.$$

Now we determine  $\lambda_i := (\partial \Delta^1 F(\mathbf{A}_0, T_1, T_2) / \partial T_i)(\mathbf{a}_0, \boldsymbol{\alpha})$  for  $i = 1, 2$ . Observe that

$$\frac{\partial}{\partial T_2} \left( \frac{T_2^j - T_1^j}{T_2 - T_1} \right) = \frac{j T_2^{j-1} (T_2 - T_1) - (T_2^j - T_1^j)}{(T_2 - T_1)^2} = \sum_{k=2}^j \binom{j}{k} T_2^{j-k} (T_1 - T_2)^{k-2}.$$

It follows that

$$\lambda_2 := \frac{\partial \Delta^1 F(\mathbf{A}_0, T_1, T_2)}{\partial T_2}(\mathbf{a}_0, \boldsymbol{\alpha}) = \sum_{j=2}^d a_j \frac{j(j-1)}{2} \alpha_1^{j-2} = \Delta^2 f_{\mathbf{a}_0}(\alpha_1, \alpha_1).$$

Furthermore, it is easy to see that  $\lambda_1 = -\lambda_2$ .

Recall that  $\alpha_i$  is a simple root of  $f_{\mathbf{a}_0}$  for  $i \geq 3$ , which implies  $f'_{\mathbf{a}_0}(\alpha_i) \neq 0$  for  $i \geq 3$ . If  $\Delta^2 f_{\mathbf{a}_0}(\alpha_1, \alpha_1) \neq 0$ , then there exists an  $(r \times r)$ -submatrix of  $(\partial(\Delta^1, \mathbf{F}^*) / \partial(\mathbf{A}_0, \mathbf{T}))(\mathbf{a}_0, \boldsymbol{\alpha})$  with rank  $r$ . Thus,  $J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha})$  has rank  $r + m$ , namely the first assertion of the claim holds. It follows that the kernel of  $J_{\mathbf{G}, \Delta^1, \mathbf{F}^*}(\mathbf{a}_0, \boldsymbol{\alpha})$  has dimension  $d - m$ . This implies that  $\dim \mathcal{T}_{(\mathbf{a}_0, \boldsymbol{\alpha})} \Gamma_r^* \leq d - m$ , which proves that  $(\mathbf{a}_0, \boldsymbol{\alpha})$  is regular point of  $\Gamma_r^*$ .

On the other hand, for a point  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \Gamma_r^*$  as in the statement of the lemma with  $\Delta^2 f_{\mathbf{a}_0}(\alpha_1, \alpha_1) = 0$ , we have that  $\alpha_1$  is root of multiplicity at least three of  $f_{\mathbf{a}_0}$ . As a consequence, the set  $\mathcal{W}_3$  of such points is contained in  $\Psi_r^{-1}(\mathcal{D}(V) \cap \mathcal{S}_1(V))$ . The lemma follows arguing as in the proof of Lemma 4.7.  $\square$

Finally, we analyze the set of points of  $\Gamma_r^*$  such that the value of at least three distinct coordinates of  $\boldsymbol{\alpha}$  is the same multiple root of  $f_{\mathbf{a}_0}$ .

**Lemma 4.9.** *Let  $\mathcal{W}_4 \subset \Gamma_r^*$  be the set of points  $(\mathbf{a}_0, \boldsymbol{\alpha})$  for which there exist  $1 \leq i < j < k \leq r$  such that  $\alpha_i = \alpha_j = \alpha_k$  and  $\alpha_i$  is a multiple root of  $f_{\mathbf{a}_0}$ . Then  $\mathcal{W}_4$  is contained in a subvariety of codimension 2 in  $\Gamma_r^*$ .*

*Proof.* Let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be an arbitrary point of  $\mathcal{W}_4$ . Without loss of generality we may assume that  $\alpha_1 = \alpha_2 = \alpha_3$  is the multiple root of  $f_{\mathbf{a}_0}$  of the statement of the lemma. Since  $(\mathbf{a}_0, \boldsymbol{\alpha})$  satisfies the equations

$$F(\mathbf{A}_0, T_1) = \Delta F(\mathbf{A}_0, T_1, T_2) = \Delta^2 F(\mathbf{A}_0, T_1, T_2, T_3) = 0,$$

we see that  $\alpha_1$  is a common root of  $f_{\mathbf{a}_0}$ ,  $\Delta F(\mathbf{a}_0, T, T)$  and  $\Delta^2 F(\mathbf{a}_0, T, T, T)$ . It follows that  $\alpha_1$  is a root of multiplicity at least 3 of  $f_{\mathbf{a}_0}$ , and thus the greatest common divisor of  $f_{\mathbf{a}_0}$  and  $f'_{\mathbf{a}_0}$  has degree at least 2. As a consequence, the proof follows by the arguments of the proof of Lemma 4.7.  $\square$

Now we can prove the main result of this section. According to Lemmas 4.5, 4.6, 4.7, 4.8 and 4.9, the set of singular points of  $\Gamma_r^*$  is contained in the set  $\mathcal{W}_1 \cup \mathcal{W}_2 \cup \mathcal{W}_3 \cup \mathcal{W}_4$ , where  $\mathcal{W}_1$ ,  $\mathcal{W}_2$ ,  $\mathcal{W}_3$  and  $\mathcal{W}_4$  are defined in the statement of Lemmas 4.5, 4.7, 4.8 and 4.9. Since each set  $\mathcal{W}_i$  is contained in a subvariety of codimension 2 of  $\Gamma_r^*$ , we obtain the following result.

**Theorem 4.10.** *Let  $q > d \geq m + 2$ . The singular locus of  $\Gamma_r^*$  has codimension at least 2 in  $\Gamma_r^*$ .*

In the next result we deduce important consequences of Theorem 4.10.

**Corollary 4.11.** *With assumptions as in Theorem 4.10, the ideal  $J \subset \mathbb{F}_q[\mathbf{A}_0, \mathbf{T}]$  generated by  $G_1, \dots, G_m$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ) is radical. Moreover,  $\Gamma_r^*$  is an ideal-theoretic complete intersection of dimension  $d - m$ .*

*Proof.* We prove that  $J$  is a radical ideal. Denote by  $J_{\mathbf{G}, \Delta}(\mathbf{A}_0, \mathbf{T})$  the Jacobian matrix of  $G_1, \dots, G_m$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ) with respect to  $\mathbf{A}_0, \mathbf{T}$ . By Lemma 4.1, these polynomials form a regular sequence. Hence, according to [Eis95, Theorem 18.15], it is sufficient to prove that the set of points  $(\mathbf{a}_0, \alpha) \in \Gamma_r^*$  for which  $J_{\mathbf{G}, \Delta}(\mathbf{a}_0, \alpha)$  does not have full rank is contained in a subvariety of  $\Gamma_r^*$  of codimension at least 1.

In the proof of Lemma 4.2 we show that  $F(\mathbf{A}_0, T_i) \in J$  for  $1 \leq i \leq r$ . This implies that each gradient  $\nabla F(\mathbf{a}_0, \alpha_i)$  is a linear combination of the gradients of  $G_1, \dots, G_m$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ). We conclude that  $\text{rank } J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \alpha) \leq \text{rank } J_{\mathbf{G}, \Delta}(\mathbf{a}_0, \alpha)$ .

Let  $(\mathbf{a}_0, \alpha)$  be an arbitrary point of  $\Gamma_r^*$  such that  $J_{\mathbf{G}, \Delta}(\mathbf{a}_0, \alpha)$  does not have full rank. By Corollary 4.4 we may assume without loss of generality that  $\mathbf{a}_0 \in V \setminus \text{Sing}(V)$ . Then  $J_{\mathbf{G}, \mathbf{F}}(\mathbf{a}_0, \alpha)$  does not have full rank and thus  $f_{\mathbf{a}_0}$  has multiple roots. Observe that the set of points  $(\mathbf{a}_0, \alpha) \in \Gamma_r^*$  for which  $f_{\mathbf{a}_0}$  has multiple roots is equal to  $\Psi_r^{-1}(\mathcal{D}(V))$ , where  $\mathcal{D}(V)$  is discriminant locus of  $V$ . According to hypothesis  $(H_4)$ ,  $\mathcal{D}(V)$  has codimension one in  $V$ , which implies that  $\Psi_r^{-1}(\mathcal{D}(V))$  has codimension 1 of  $\Gamma_r^*$ . It follows that the set of points  $(\mathbf{a}_0, \alpha) \in \Gamma_r^*$  for which  $J_{\mathbf{G}, \Delta}(\mathbf{a}_0, \alpha)$  does not have full rank is contained in a subvariety of  $\Gamma_r^*$  of codimension at least 1. Hence,  $J$  is a radical ideal, which in turn implies that  $\Gamma_r^*$  is an ideal-theoretic complete intersection of dimension  $d - m$ .  $\square$

## 5. THE GEOMETRY OF THE PROJECTIVE CLOSURE OF $\Gamma_r^*$

To estimate the number of  $\mathbb{F}_q$ -rational points of  $\Gamma_r^*$  we need information on the behavior of  $\Gamma_r^*$  at infinity. For this purpose, we shall analyze the projective closure of  $\Gamma_r^*$ , whose definition we now recall. Consider the embedding of  $\mathbb{A}^{d+r}$  into the projective space  $\mathbb{P}^{d+r}$  which assigns to any point  $(\mathbf{a}_0, \alpha) \in \mathbb{A}^{d+r}$  the point  $(a_{d-1} : \dots : a_0 : 1 : \alpha_1 : \dots : \alpha_r) \in \mathbb{P}^{d+r}$ . The closure in the Zariski topology of  $\mathbb{P}^{d+r}$  of the image of  $\Gamma_r^*$  under this embedding is called the *projective closure*  $\text{pcl}(\Gamma_r^*) \subset \mathbb{P}^{d+r}$  of  $\Gamma_r^*$ . The points of  $\text{pcl}(\Gamma_r^*)$  lying in  $\{T_0 = 0\}$  are called the points of  $\text{pcl}(\Gamma_r^*)$  *at infinity*.

It is well-known that  $\text{pcl}(\Gamma_r^*)$  is the  $\mathbb{F}_q$ -variety of  $\mathbb{P}^{d+r}$  defined by the homogenization  $F^h \in \mathbb{F}_q[\mathbf{A}_0, T_0, \mathbf{T}]$  of each polynomial  $F$  in the ideal  $J \subset \mathbb{F}_q[\mathbf{A}_0, \mathbf{T}]$  generated by  $G_1, \dots, G_m$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ). We denote by  $J^h$  the ideal generated by all the polynomials  $F^h$  with  $F \in J$ . Since  $J$  is radical it turns out that  $J^h$  is also radical (see, e.g., [Kun85, §I.5, Exercise 6]). Furthermore,  $\text{pcl}(\Gamma_r^*)$  is of pure dimension  $d - m$  (see, e.g., [Kun85, Propositions I.5.17 and II.4.1]) and degree equal to  $\deg \Gamma_r^*$  (see, e.g., [CGH91, Proposition 1.11]).

**Lemma 5.1.** *The polynomials  $G_1^h, \dots, G_m^h$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)^h$  ( $1 \leq i \leq r$ ) form a regular sequence of  $\mathbb{F}_q[\mathbf{A}_0, T_0, \mathbf{T}]$ .*

*Proof.* We claim  $G_1^h, \dots, G_m^h$  form a regular sequence of  $\mathbb{F}_q[\mathbf{A}_0, T_0, \mathbf{T}]$ . Indeed, the projective subvariety of  $\mathbb{P}^{d+r}$  defined by  $T_0$  and  $G_1^h, \dots, G_m^h$  is isomorphic to the subvariety of  $\mathbb{P}^{d+r-1}$  defined by  $G_1^{d_1}, \dots, G_m^{d_m}$ . Since hypothesis  $(H_3)$  implies that the latter is of pure dimension  $d + r - m - 1$ , we conclude that the subvariety of  $\mathbb{P}^{d+r}$  defined by  $G_1^h, \dots, G_m^h$  is of pure dimension  $d + r - m$ . It follows that  $G_1^h, \dots, G_m^h$  form a regular sequence. Then the lemma follows arguing as in the proof of Lemma 4.1.  $\square$

**Proposition 5.2.** *The projective variety defined by  $G_1^h, \dots, G_m^h$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)^h$  ( $1 \leq i \leq r$ ) is  $\text{pcl}(\Gamma_r^*)$ . Therefore,  $\text{pcl}(\Gamma_r^*)$  is a set-theoretic complete intersection of dimension  $d - m$ .*

*Proof.* By the Newton form of the polynomial interpolating  $F(\mathbf{A}_0, T)$  at the points  $T_1, \dots, T_r$  we conclude that

$$F(\mathbf{A}_0, T_j) = \sum_{i=1}^r \Delta^{i-1} F(\mathbf{A}_0, T_1, \dots, T_i) (T_j - T_1) \cdots (T_j - T_{i-1}).$$

Homogenizing both sides of this equality we deduce that  $F(\mathbf{A}_0, T_j)^h$  belongs to the ideal of  $\mathbb{F}_q[\mathbf{A}_0, T_0, \mathbf{T}]$  generated by  $G_1^h, \dots, G_m^h$  and  $\Delta^{i-1} F(\mathbf{A}_0, T_1, \dots, T_i)^h$ . Denote by  $V^h \subset \mathbb{P}^{d+r}$  the variety defined by all these polynomials. Any point of  $V^h \cap \{T_0 = 0\} \subset \mathbb{P}^{d+r}$  satisfies

$$\begin{aligned} F(\mathbf{A}_0, T_i)^h|_{T_0=0} &= T_i^d + A_{d-1} T_i^{d-1} = T_i^{d-1} (T_i + A_{d-1}) = 0 \quad (1 \leq i \leq r), \\ G_k^h(\mathbf{A}, T_0)^h|_{T_0=0} &= G_k^{d_k}(\mathbf{A}) = 0 \quad (1 \leq k \leq m). \end{aligned}$$

We deduce that  $V^h \cap \{T_0 = 0\}$  is contained in the union

$$V^h \cap \{T_0 = 0\} \subset \bigcup_{\mathcal{I} \subset \{1, \dots, r\}} V_{\mathcal{I}} \cap \{T_0 = 0\},$$

where  $V_{\mathcal{I}} \subset \mathbb{P}^{d+r}$  is the variety defined by  $T_i = 0$  ( $i \in \mathcal{I}$ ),  $T_j + A_{d-1} = 0$  ( $j \in \{1, \dots, r\} \setminus \mathcal{I}$ ) and  $G_k^{d_k} = 0$  ( $1 \leq k \leq m$ ). As any  $V_{\mathcal{I}} \cap \{T_0 = 0\}$  is of pure dimension  $d - m - 1$ ,  $V^h \cap \{T_0 = 0\}$  is of pure dimension  $d - m - 1$ .

Lemma 5.1 implies that  $V^h$  is of pure dimension  $d - m$ , and thus it has no irreducible component in the hyperplane at infinity. In particular, it agrees with the projective closure of its restriction to  $\mathbb{A}^{d+r}$  (see, e.g., [Kun85, Proposition I.5.17]). As this restriction is  $\Gamma_r^*$ , we have  $V^h = \text{pcl}(\Gamma_r^*)$ .  $\square$

**5.1. The singular locus of  $\text{pcl}(\Gamma_r^*)$ .** Next we study the singular locus of  $\text{pcl}(\Gamma_r^*)$ . We start with the following characterization of the points of  $\text{pcl}(\Gamma_r^*)$  at infinity.

**Lemma 5.3.**  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\} \subset \mathbb{P}^{d+r-1}$  is contained in a union of  $r+1$  normal complete intersections defined over  $\mathbb{F}_q$ , each of pure dimension  $d - m - 1$  and degree  $\prod_{i=1}^m d_i$ .

*Proof.* We claim that  $\Delta^1 F(\mathbf{A}_0, T_i, T_j)^h \in J^h$  for  $1 \leq i < j \leq r$ . Indeed, we have the identity  $\Delta^1 F(\mathbf{A}_0, T_i, T_j)(T_i - T_j) = F(\mathbf{A}_0, T_i) - F(\mathbf{A}_0, T_j)$ . Since  $F(\mathbf{A}_0, T_k)$  vanishes in  $\Gamma_r^*$  for  $1 \leq k \leq r$ , we deduce that  $\Delta^1 F(\mathbf{A}_0, T_i, T_j)$  vanishes on the nonempty Zariski open dense subset  $\{T_i \neq T_j\} \cap \Gamma_r^*$  of  $\Gamma_r^*$ . This implies that  $\Delta^1 F(\mathbf{A}_0, T_i, T_j)$  vanishes in  $\Gamma_r^*$ , which proves the claim.

Combining the claim with the fact that  $F(\mathbf{A}_0, T_i)^h \in J^h$  for  $1 \leq i \leq r$ , we conclude that any  $(\mathbf{a}_0, \boldsymbol{\alpha}) \in \text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$  satisfies the following identities for  $1 \leq i \leq r$  and  $1 \leq i < j \leq r$  respectively:

$$(5.1) \quad F(\mathbf{A}_0, T_i)^h|_{T_0=0} = T_i^d + A_{d-1} T_i^{d-1} = T_i^{d-1} (T_i + A_{d-1}) = 0,$$

$$\begin{aligned} \Delta^1 F(\mathbf{A}_0, T_i, T_j)^h|_{T_0=0} &= \frac{T_i^d - T_j^d}{T_i - T_j} + A_{d-1} \frac{T_i^{d-1} - T_j^{d-1}}{T_i - T_j} \\ (5.2) \quad &= \sum_{k=0}^{d-2} T_j^k T_i^{d-2-k} (T_i + A_{d-1}) + T_j^{d-1} = 0. \end{aligned}$$

From (5.1)–(5.2) we deduce that  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$  is contained in a finite union of  $r+1$  normal complete intersections of  $\mathbb{P}^{d+r-1}$  defined over  $\mathbb{F}_q$  of pure dimension  $d - m - 1$ . More precisely, it can be seen that

$$\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\} \subset \bigcup_{i=0}^r V_i \cap \{T_0 = 0\},$$

where  $V_0$  is the variety defined by  $T_i = 0$  ( $1 \leq i \leq r$ ) and  $G_k^{d_k} = 0$  ( $1 \leq k \leq m$ ), and  $V_i$  ( $1 \leq i \leq r$ ) is defined as the set of solutions of

$$T_i + A_{d-1} = 0, \quad T_j = 0 \quad (1 \leq j \leq r, i \neq j), \quad G_k^{d_k} = 0 \quad (1 \leq k \leq m).$$

By Proposition 5.2 we have that  $\text{pcl}(\Gamma_r^*)$  is of pure dimension  $d-m$ . Then each irreducible component  $\mathcal{C}$  of  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$  has dimension at least  $d-m-1$ , and is contained in an irreducible component of a variety  $V_i$  for some  $i \in \{0, \dots, r\}$ . By, e.g., [Sha94, §6.1, Theorem 1],  $\mathcal{C}$  is an irreducible component of a variety  $V_i$ , finishing thus the proof of the lemma.  $\square$

Now we are able to upper bound the dimension of the singular locus of  $\text{pcl}(\Gamma_r^*)$  at infinity.

**Lemma 5.4.** *The singular locus of  $\text{pcl}(\Gamma_r^*)$  at infinity has dimension at most  $d-m-2$ .*

*Proof.* By [GL02, Lemma 1.1], the singular locus of  $\text{pcl}(\Gamma_r^*)$  at infinity is contained in the singular locus of  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$ . Lemma 5.3 proves that  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$  has pure dimension  $d-m-1$ . Therefore, its singular locus has dimension at most  $d-m-2$ .  $\square$

**Lemma 5.5.** *The polynomials  $G_1^h, \dots, G_m^h$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)^h$  ( $1 \leq i \leq r$ ) generate  $J^h$ . Hence,  $\text{pcl}(\Gamma_r^*)$  is an ideal-theoretic complete intersection of dimension  $d-m$  and multidegree  $(d_1, \dots, d_m, d, \dots, d-r+1)$ .*

*Proof.* According to [Eis95, Theorem 18.15], it suffices to prove that of the set of points of  $\text{pcl}(\Gamma_r^*)$  for which the Jacobian of  $G_1^h, \dots, G_m^h$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)^h$  ( $1 \leq i \leq r$ ) does not have full rank, has codimension at least one in  $\text{pcl}(\Gamma_r^*)$ . The set of points of  $\{T_0 \neq 0\}$  for which such a Jacobian does not have full rank, has codimension one, because  $G_1, \dots, G_m$  and  $\Delta^{i-1}F(\mathbf{A}_0, T_1, \dots, T_i)$  ( $1 \leq i \leq r$ ) define a radical ideal. On the other hand, the set of points of  $\text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\}$  has codimension one in  $\text{pcl}(\Gamma_r^*)$ . This proves the first assertion of the lemma.

We deduce that  $\text{pcl}(\Gamma_r^*)$  is an ideal-theoretic complete intersection of dimension  $d-m$ , and the Bézout theorem proves that  $\deg \text{pcl}(\Gamma_r^*) = \prod_{i=1}^m d_i \cdot d! / (d-r)!$ .  $\square$

Finally, we prove the main result of this section.

**Theorem 5.6.** *For  $q > d \geq m+2$ ,  $\text{pcl}(\Gamma_r^*) \subset \mathbb{P}^{d+r}$  is a normal ideal-theoretic complete intersection of dimension  $d-m$  and multidegree  $(d_1, \dots, d_m, d, \dots, d-r+1)$ .*

*Proof.* Lemma 5.5 shows that  $\text{pcl}(\Gamma_r^*)$  is an ideal-theoretic complete intersection of dimension  $d-m$  and multidegree  $(d_1, \dots, d_m, d, \dots, d-r+1)$ . On the other hand, Theorem 4.10 and Lemma 5.4 show that the singular locus of  $\text{pcl}(\Gamma_r^*)$  has codimension at least 2 in  $\text{pcl}(\Gamma_r^*)$ . This implies  $\text{pcl}(\Gamma_r^*)$  is regular in codimension 1 and thus normal.  $\square$

By Theorems 2.1 and 5.6 we conclude that  $\text{pcl}(\Gamma_r^*)$  is absolutely irreducible of dimension  $d-m$ , and the same holds for  $\Gamma_r^* \subset \mathbb{A}^{d+r}$ . Since  $\Gamma_r$  is a nonempty Zariski open subset of  $\Gamma_r^*$  of dimension  $d-m$  and  $\Gamma_r^*$  is absolutely irreducible, the Zariski closure of  $\Gamma_r$  is  $\Gamma_r^*$ .

## 6. THE NUMBER OF $\mathbb{F}_q$ -RATIONAL POINTS OF $\Gamma_r$

As before, let  $d, m$  be positive integers with  $q > d \geq m+2$ . Let  $A_{d-1}, \dots, A_1$  be indeterminates over  $\mathbb{F}_q$ , set  $\mathbf{A} := (A_{d-1}, \dots, A_1)$ , and let  $G_1, \dots, G_m$  be polynomials of  $\mathbb{F}_q[A_{d-1}, \dots, A_1]$  satisfying hypotheses  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_4)$ . Let  $\mathbf{G} := (G_1, \dots, G_m)$  and  $\mathcal{A} := \mathcal{A}(\mathbf{G})$  be the family defined as

$$\mathcal{A} := \{T^d + a_{d-1}T^{d-1} + \dots + a_1T \in \mathbb{F}_q[T] : \mathbf{G}(a_{d-1}, \dots, a_1) = \mathbf{0}\}.$$

In this section we determine the asymptotic behavior of the average value set  $\mathcal{V}(\mathcal{A})$  of  $\mathcal{A}$ . By Lemma 3.1 we have

$$\mathcal{V}(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |S_{\mathcal{X}_r}^{\mathcal{A}}|,$$



where the second sum runs through all the subsets  $\mathcal{X}_r \subset \mathbb{F}_q$  of cardinality  $r$  and  $S_{\mathcal{X}_r}^A$  denotes the number of  $f + a_0 \in \mathcal{A} + \mathbb{F}_q$  such that  $(f + a_0)(\alpha) = 0$  for any  $\alpha \in \mathcal{X}_r$ .

Let  $S_r^A := \sum_{\mathcal{X}_r \subset \mathbb{F}_q} |S_{\mathcal{X}_r}^A|$ . According to Lemmas 3.2 and 3.3, for  $1 \leq r \leq d$  we have

$$S_r^A = \frac{|\Gamma_r(\mathbb{F}_q)|}{r!} = \frac{1}{r!} \left| \Gamma_r^*(\mathbb{F}_q) \setminus \bigcup_{i \neq j} \{T_i = T_j\} \right|.$$

We shall apply the results on the geometry of  $\Gamma_r^*$  of the previous section in order to estimate the number of  $\mathbb{F}_q$ -rational points of  $\Gamma_r^*$ .

**6.1. An estimate for  $S_r^A$ .** We shall rely on the following estimate ([CMP15a, Theorem 1.3]; see also [GL02], [CM07] or [MPP16] for similar estimates): if  $W \subset \mathbb{P}^n$  is a normal complete intersection defined over  $\mathbb{F}_q$  of dimension  $l \geq 2$  and multidegree  $(e_1, \dots, e_{n-l})$ , then

$$(6.1) \quad ||W(\mathbb{F}_q)| - p_l| \leq (\delta_W(D_W - 2) + 2)q^{l-\frac{1}{2}} + 14D_W^2\delta_W^2q^{l-1},$$

where  $p_l := q^l + q^{l-1} + \dots + 1$ ,  $\delta_W := \prod_{i=1}^{n-l} e_i$  and  $D_W := \sum_{i=1}^{n-l} (e_i - 1)$ .

In what follows, we shall use the following notations:

$$\begin{aligned} \delta_V &:= \prod_{i=1}^m d_i, & \delta_{\Delta_r} &:= \prod_{i=1}^r (d - i + 1) = \frac{d!}{(d-r)!}, & \delta_r &:= \delta_V \delta_{\Delta_r}, \\ D_V &:= \sum_{i=1}^m (d_i - 1), & D_{\Delta_r} &:= \sum_{i=1}^r (d - i) = rd - \frac{r(r+1)}{2}, & D_r &:= D_V + D_{\Delta_r}. \end{aligned}$$

We start with an estimate on the number of elements of  $\mathcal{A}$ .

**Lemma 6.1.** *For  $q > 16(D_V \delta_V + 14D_V^2 \delta_V^2 q^{-\frac{1}{2}})^2$ , we have*

$$\frac{1}{2}q^{d-m-1} < |\mathcal{A}| \leq q^{d-m-1} + 2(\delta_V(D_V - 2) + 2 + 14D_V^2 \delta_V^2 q^{-\frac{1}{2}})q^{d-m-\frac{3}{2}}.$$

*Proof.* Hypothesis (H<sub>1</sub>), (H<sub>2</sub>) and (H<sub>3</sub>) imply that both the projective closure  $\text{pcl}(V) \subset \mathbb{P}^{d-1}$  of  $V$  and the set  $\text{pcl}(V)^\infty := \text{pcl}(V) \cap \{T_0 = 0\}$  of points at infinity are normal complete intersections defined over  $\mathbb{F}_q$ , both of multidegree  $(d_1, \dots, d_m)$  in  $\mathbb{P}^{d-1}$  and  $\{T_0 = 0\} \cong \mathbb{P}^{d-2}$  respectively. Therefore, by (6.1) it follows that

$$\begin{aligned} ||\mathcal{A}| - q^{d-m-1}| &= ||\text{pcl}(V)(\mathbb{F}_q)| - |\text{pcl}(V)^\infty(\mathbb{F}_q)| - p_{d-m-1} + p_{d-m-2}| \\ &\leq ||\text{pcl}(V)(\mathbb{F}_q)| - p_{d-m-1}| + ||\text{pcl}(V)^\infty(\mathbb{F}_q)| - p_{d-m-2}| \\ &\leq (\delta_V(D_V - 2) + 2)(q + 1)q^{d-m-\frac{5}{2}} + 14D_V^2 \delta_V^2 (q + 1)q^{d-m-3} \\ &\leq 2(\delta_V(D_V - 2) + 2 + 14D_V^2 \delta_V^2 q^{-\frac{1}{2}})q^{d-m-\frac{3}{2}}. \end{aligned}$$

By the hypothesis on  $q$  of the statement, the lemma readily follows.  $\square$

By Theorem 5.6,  $\text{pcl}(\Gamma_r^*) \subset \mathbb{P}^{d+r}$  is a normal complete intersection defined over  $\mathbb{F}_q$  of dimension  $d - m$  and multidegree  $(d_1, \dots, d_m, d, \dots, d - r + 1)$ . Therefore, applying (6.1) we obtain

$$||\text{pcl}(\Gamma_r^*)(\mathbb{F}_q)| - p_{d-m}| \leq (\delta_r(D_r - 2) + 2)q^{d-m-\frac{1}{2}} + 14D_r^2 \delta_r^2 q^{d-m-1}.$$

On the other hand, since  $\text{pcl}(\Gamma_r^*)^\infty := \text{pcl}(\Gamma_r^*) \cap \{T_0 = 0\} \subset \mathbb{P}^{d+r-1}$  is a finite union of at most  $r + 1$  varieties, each of pure dimension  $d - m - 1$  and degree  $\delta_V$ , by (2.3) we have  $|\text{pcl}(\Gamma_r^*)^\infty(\mathbb{F}_q)| \leq (r + 1)\delta_V p_{d-m-1}$ . Hence,

$$\begin{aligned} (6.2) \quad ||\Gamma_r^*(\mathbb{F}_q)| - q^{d-m}| &\leq ||\text{pcl}(\Gamma_r^*)(\mathbb{F}_q)| - p_{d-m}| + ||\text{pcl}(\Gamma_r^*)(\mathbb{F}_q)^\infty| - p_{d-m-1}| \\ &\leq (\delta_r(D_r - 2) + 2)q^{d-m-\frac{1}{2}} + 14D_r^2 \delta_r^2 q^{d-m-1} + (r + 1)\delta_V p_{d-m-1} \\ &\leq (\delta_r(D_r - 2) + 2)q^{d-m-\frac{1}{2}} + (14D_r^2 \delta_r^2 + 4r\delta_V)q^{d-m-1}. \end{aligned}$$

We also need an upper bound on the number  $\mathbb{F}_q$ -rational points of

$$\Gamma_r^{*,=} := \Gamma_r^* \bigcap \bigcup_{1 \leq i < j \leq r} \{T_i = T_j\}.$$

We observe that  $\Gamma_r^{*,=} = \Gamma_r^* \cap \mathcal{H}_r$ , where  $\mathcal{H}_r \subset \mathbb{A}^{d+r}$  is the hypersurface defined by the polynomial  $F_r := \prod_{1 \leq i < j \leq r} (T_i - T_j)$ . From the Bézout inequality (2.1) it follows that

$$(6.3) \quad \deg \Gamma_r^{*,=} \leq \delta_r \binom{r}{2}.$$

Furthermore, we claim that  $\Gamma_r^{*,=}$  has dimension at most  $d - m - 1$ . Indeed, let  $(\mathbf{a}_0, \boldsymbol{\alpha})$  be any point of  $\Gamma_r^{*,=}$ . Assume without loss of generality that  $\alpha_1 = \alpha_2$ . By the definition of divided differences we deduce that  $f'_{\mathbf{a}_0}(\alpha_1) = 0$ , which implies that  $f_{\mathbf{a}_0}$  has multiple roots. By the proof of Corollary 4.11, the set of points  $(\mathbf{a}_0, \boldsymbol{\alpha})$  of  $\Gamma_r^*$  for which  $f_{\mathbf{a}_0}$  has multiple roots is contained in a subvariety of  $\Gamma_r^*$  of codimension 1, which proves the claim.

Combining the claim with (6.3) and (2.2), we obtain

$$(6.4) \quad |\Gamma_r^{*,=}(\mathbb{F}_q)| \leq \delta_r \binom{r}{2} q^{d-m-1}.$$

Since  $\Gamma_r(\mathbb{F}_q) = \Gamma_r^*(\mathbb{F}_q) \setminus \Gamma_r^{*,=}(\mathbb{F}_q)$ , from (6.2) and (6.4) we deduce that

$$\begin{aligned} \left| |\Gamma_r(\mathbb{F}_q)| - q^{d-m} \right| &\leq \left| |\Gamma_r^*(\mathbb{F}_q)| - q^{d-m} \right| + |\Gamma_r^{*,=}(\mathbb{F}_q)| \\ &\leq (\delta_r(D_r - 2) + 2) q^{d-m-\frac{1}{2}} + \left( 14D_r^2\delta_r^2 + \binom{r}{2}\delta_r + 4r\delta_V \right) q^{d-m-1}. \end{aligned}$$

As a consequence, we obtain the following result.

**Theorem 6.2.** *Let  $q > d \geq m + 2$ . For any  $r$  with  $1 \leq r \leq d$ , we have*

$$\left| S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right| \leq \left( \frac{\delta_r(D_r - 2) + 2}{r!} q^{\frac{1}{2}} + \left( 14 \frac{D_r^2\delta_r^2}{r!} + \binom{r}{2} \frac{\delta_r}{r!} + \frac{4r}{r!} \delta_V \right) \right) q^{d-m-1}.$$

**6.2. An estimate for the average value set  $\mathcal{V}(\mathcal{A})$ .** Theorem 6.2 is the critical step in our approach to estimate  $\mathcal{V}(\mathcal{A})$ .

**Corollary 6.3.** *With assumptions as in Lemma 6.1 and Theorem 6.2,*

$$(6.5) \quad |\mathcal{V}(\mathcal{A}) - \mu_d q| \leq 2^d \delta_V (3D_V + d^2) q^{1/2} + \frac{7}{4} \delta_V^2 D_V^2 d^4 \sum_{k=0}^{d-1} \binom{d}{k}^2 (d-k)!.$$

*Proof.* According to Lemma 3.1, we have

$$\begin{aligned} \mathcal{V}(\mathcal{A}) - \mu_d q &= \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \left( S_r^{\mathcal{A}} - \frac{|\mathcal{A}|q}{r!} \right) \\ &= \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \left( S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right) - \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \left( \frac{|\mathcal{A}|q - q^{d-m}}{r!} \right) \\ (6.6) \quad &= \frac{1}{|\mathcal{A}|} \sum_{r=1}^d (-1)^{r-1} \left( S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right) + \mu_d \left( \frac{q^{d-m} - |\mathcal{A}|q}{|\mathcal{A}|} \right). \end{aligned}$$

We consider the absolute value of the first sum in the right-hand side of (6.6). From Lemma 6.1 and Theorem 6.2 we have

$$\begin{aligned} \frac{1}{|\mathcal{A}|} \sum_{r=1}^d \left| S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right| &\leq \frac{2}{q^{d-m-1}} \sum_{r=1}^d \left| S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right| \\ &\leq 2q^{\frac{1}{2}} \sum_{r=1}^d \frac{\delta_r(D_r - 2) + 2}{r!} + 28 \sum_{r=1}^d \frac{D_r^2 \delta_r^2}{r!} + 2\delta_V \sum_{r=1}^d \frac{\binom{r}{2} \delta_{\Delta_r} + 4r}{r!}. \end{aligned}$$

Concerning the first sum in the right-hand side, we see that

$$\begin{aligned} \sum_{r=1}^d \frac{\delta_r(D_r - 2) + 2}{r!} &\leq \delta_V \left( D_V \sum_{r=1}^d \binom{d}{r} + \sum_{r=1}^d \frac{\delta_{\Delta_r}(D_{\Delta_r} - 2) + 2}{r!} \right) \\ &\leq \delta_V (D_V 2^d + d^2 2^{d-1}) = 2^{d-1} \delta_V (2D_V + d^2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \sum_{r=1}^d \frac{D_r^2 \delta_r^2}{r!} &= \delta_V^2 \left( D_V^2 \sum_{r=1}^d \frac{\delta_{\Delta_r}^2}{r!} + 2D_V \sum_{r=1}^d \frac{D_{\Delta_r} \delta_{\Delta_r}^2}{r!} + \sum_{r=1}^d \frac{D_{\Delta_r}^2 \delta_{\Delta_r}^2}{r!} \right) \\ &\leq \delta_V^2 \left( \frac{D_V^2}{4} + D_V + 1 \right) \sum_{r=1}^d \frac{D_{\Delta_r}^2 \delta_{\Delta_r}^2}{r!} \\ &\leq \delta_V^2 \frac{(D_V^2 + 2)^2}{4} \frac{1}{64} (2d - 1)^4 \sum_{k=0}^{d-1} \binom{d}{k}^2 (d - k)!. \end{aligned}$$

Finally, we consider the last sum

$$\sum_{r=1}^d \frac{\delta_{\Delta_r}}{2(r-2)!} = \sum_{r=1}^d \binom{d}{r} \frac{r(r-1)}{2} = \sum_{k=0}^{d-1} \binom{d}{k} \frac{(d-k)!}{2(d-k-2)!}.$$

As a consequence, we obtain

$$\begin{aligned} \frac{1}{|\mathcal{A}|} \sum_{r=1}^d \left| S_r^{\mathcal{A}} - \frac{q^{d-m}}{r!} \right| &\leq q^{\frac{1}{2}} 2^d \delta_V (2D_V + d^2) + \frac{7}{64} (2d - 1)^4 \sum_{k=0}^{d-1} \binom{d}{k}^2 (d - k)! \\ &\quad + \sum_{k=0}^{d-1} \binom{d}{k} (d - k)! + 8 \sum_{k=0}^{d-1} \frac{1}{(d - k - 1)!}. \end{aligned}$$

Concerning the second sum in the right-hand side of (6.6), by Lemma 6.1 it follows that

$$\left| \frac{q^{d-m} - |\mathcal{A}|q}{|\mathcal{A}|} \right| \leq 4(\delta_V(D_V - 2) + 2 + 14D_V^2 \delta_V^2 q^{-\frac{1}{2}}) q^{\frac{1}{2}}.$$

The statement of the corollary follows by elementary calculations.  $\square$

Next we analyze the behavior of the right-hand side of (6.5). This analysis consists of elementary calculations, which are only sketched.

Fix  $k$  with  $0 \leq k \leq d - 1$  and denote  $h(k) := \binom{d}{k}^2 (d - k)!$ . From an analysis of the sign of the differences  $h(k + 1) - h(k)$  for  $0 \leq k \leq d - 1$  we deduce the following remark, which is stated without proof.

**Remark 6.4.** Let  $k_0 := -1/2 + \sqrt{5 + 4d}/2$ . Then  $h$  is either an increasing function or a unimodal function in the integer interval  $[0, d - 1]$ , which reaches its maximum at  $\lfloor k_0 \rfloor$ .

From Remark 6.4 we see that

$$(6.7) \quad \sum_{k=0}^{d-1} \binom{d}{k}^2 (d - k)! \leq d \binom{d}{\lfloor k_0 \rfloor}^2 (d - \lfloor k_0 \rfloor)! = \frac{d(d!)^2}{(d - \lfloor k_0 \rfloor)! (\lfloor k_0 \rfloor!)^2}.$$

Now we use the following version of the Stirling formula (see, e.g., [FS09, p. 747]): for  $m \in \mathbb{N}$ , there exists  $\theta$  with  $0 \leq \theta < 1$  such that

$$m! = (m/e)^m \sqrt{2\pi m} e^{\theta/12m}.$$

By the Stirling formula there exist  $\theta_i$  ( $i = 1, 2, 3$ ) with  $0 \leq \theta_i < 1$  such that

$$C(d) := \frac{d(d!)^2}{(d - \lfloor k_0 \rfloor)! (\lfloor k_0 \rfloor!)^2} \leq \frac{d d^{2d+1} e^{-d+\lfloor k_0 \rfloor} e^{\frac{\theta_1}{6d} - \frac{\theta_2}{12(d-\lfloor k_0 \rfloor)} - \frac{\theta_3}{6\lfloor k_0 \rfloor}}}{(d - \lfloor k_0 \rfloor)^{d-\lfloor k_0 \rfloor} \sqrt{2\pi(d - \lfloor k_0 \rfloor)} [\lfloor k_0 \rfloor]^{2\lfloor k_0 \rfloor+1}}.$$

By elementary calculations we obtain

$$(d - \lfloor k_0 \rfloor)^{-d+\lfloor k_0 \rfloor} \leq d^{-d+\lfloor k_0 \rfloor} e^{\frac{\lfloor k_0 \rfloor(d-\lfloor k_0 \rfloor)}{d}}, \quad \frac{d^{\lfloor k_0 \rfloor}}{[\lfloor k_0 \rfloor]^{2\lfloor k_0 \rfloor}} \leq e^{\frac{d-\lfloor k_0 \rfloor}{\lfloor k_0 \rfloor}}.$$

It follows that

$$C(d) \leq \frac{d^{d+2} e^{2\lfloor k_0 \rfloor} e^{-\frac{\lfloor k_0 \rfloor^2}{d} + \frac{1}{6d} + \frac{d-\lfloor k_0 \rfloor}{\lfloor k_0 \rfloor}}}{\sqrt{2\pi} e^d \sqrt{d - \lfloor k_0 \rfloor} [\lfloor k_0 \rfloor]}.$$

By the definition of  $\lfloor k_0 \rfloor$ , it is easy to see that  $d/\lfloor k_0 \rfloor \sqrt{d - \lfloor k_0 \rfloor} \leq 5/2$  and that  $2\lfloor k_0 \rfloor \leq -1 + \sqrt{5 + 4d} \leq -1/5 + 2\sqrt{d}$ . Therefore, taking into account that  $d \geq 2$ , we conclude that

$$C(d) \leq \frac{5}{2} \frac{e^{\frac{109}{30}} d^{d+1} e^{2\sqrt{d}}}{\sqrt{2\pi} e^d}.$$

Combining this bound with Corollary 6.3 we obtain the following result.

**Theorem 6.5.** *For  $q > \max \{d, 16(D_V \delta_V + 14D_V^2 \delta_V^2 q^{-\frac{1}{2}})^2\}$  and  $d \geq m + 2$ , the following estimate holds:*

$$|\mathcal{V}(\mathcal{A}) - \mu_d q| \leq 2^d \delta_V (3D_V + d^2) q^{\frac{1}{2}} + 67 \delta_V^2 (D_V + 2)^2 d^{d+5} e^{2\sqrt{d}-d}.$$

**6.3. Applications of our main result.** We discuss two families of examples where hypotheses  $(H_1)$ ,  $(H_2)$ ,  $(H_3)$  and  $(H_4)$  hold. Therefore, the estimate of Theorem 6.5 is valid for these families.

Our first example concerns linear families of polynomials. Let  $L_i \in \mathbb{F}_q[A_{d-1}, \dots, A_2]$  be polynomials of degree 1 ( $1 \leq i \leq m$ ). Assume without loss of generality that the Jacobian matrix of  $L_1, \dots, L_m$  with respect to  $A_{d-1}, \dots, A_2$  is of full rank  $m \leq d - 2$ . Consider the family  $\mathcal{A}_{\mathcal{L}}$  defined as

$$\mathcal{A}_{\mathcal{L}} := \{T^d + a_{d-1}T^{d-1} + \dots + a_0 \in \mathbb{F}_q[T] : L_i(a_{d-1}, \dots, a_2) = 0 \ (1 \leq i \leq m)\}.$$

It is clear that hypotheses  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  hold. Now we analyze the validity of  $(H_4)$ . Denote by  $\mathcal{L} \subset \mathbb{A}^d$  the linear variety defined by  $L_1, \dots, L_m$ , and let  $\mathcal{D}(\mathcal{L}) \subset \mathbb{A}^d$  and  $\mathcal{S}_1(\mathcal{L}) \subset \mathbb{A}^d$  be the discriminant locus and the first subdiscriminant locus of  $\mathcal{L}$ . Since the coordinate ring  $\overline{\mathbb{F}}_q[\mathcal{L}]$  is a domain, hypothesis  $(H_4)$  holds if the coordinate function defined by the discriminant  $\text{Disc}(F(\mathbf{A}_0, T))$  in  $\overline{\mathbb{F}}_q[\mathcal{L}]$  is nonzero, and the class of the subdiscriminant  $\text{Subdisc}(F(\mathbf{A}_0, T))$  in the quotient ring  $\overline{\mathbb{F}}_q[\mathcal{L}]/\text{Disc}(F(\mathbf{A}_0, T))$  is not a zero divisor. For fields  $\mathbb{F}_q$  of characteristic  $p$  not dividing  $d(d-1)$ , both assertions are consequences of [MPP14, Theorem A.3]. Taking into account that  $\delta_{\mathcal{L}} = 1$  and  $D_{\mathcal{L}} = 0$ , applying Theorem 6.5 we obtain the following result.

**Theorem 6.6.** *For  $p := \text{char}(\mathbb{F}_q)$  not dividing  $d(d-1)$  and  $q > d \geq m + 2$ ,*

$$|\mathcal{V}(\mathcal{A}_{\mathcal{L}}) - \mu_d q| \leq 2^d d^2 q^{\frac{1}{2}} + 268 d^{d+5} e^{2\sqrt{d}-d}.$$

Our second example consists of a nonlinear family of polynomials. Let  $s, m$  be positive integers with  $m \leq s \leq d - m - 4$ , let  $\Pi_1, \dots, \Pi_s$  be the first  $s$  elementary symmetric polynomials of  $\mathbb{F}_q[A_{d-1}, \dots, A_2]$  and let  $G_1, \dots, G_m \in \mathbb{F}_q[A_{d-1}, \dots, A_2]$  be symmetric

polynomials of the form  $G_i := S_i(\Pi_1, \dots, \Pi_s)$  ( $1 \leq i \leq m$ ). Consider the weight function  $\text{wt} : \mathbb{F}_q[Y_1, \dots, Y_s] \rightarrow \mathbb{N}$  defined by setting  $\text{wt}(Y_i) := i$  ( $1 \leq i \leq s$ ) and denote by  $S_1^{\text{wt}}, \dots, S_m^{\text{wt}}$  the components of highest weight of  $S_1, \dots, S_m$ . Assume that both  $S_1, \dots, S_m$  and  $S_1^{\text{wt}}, \dots, S_m^{\text{wt}}$  form regular sequences of  $\mathbb{F}_q[Y_1, \dots, Y_s]$ , and the Jacobian matrices of  $S_1, \dots, S_m$  and  $S_1^{\text{wt}}, \dots, S_m^{\text{wt}}$  with respect to  $Y_1, \dots, Y_s$  have full rank in  $\mathbb{A}^s$ . We remark that varieties defined by polynomials of this type arise in several combinatorial problems over finite fields (see, e.g., [CMP12], [CMPP14], [MPP14], [CMP15b] and [MPP15]). Finally, let

$$\mathcal{A}_{\mathcal{N}} := \{T^d + a_{d-1}T^{d-1} + \dots + a_0 \in \mathbb{F}_q[T] : G_i(a_{d-1}, \dots, a_2) = 0 \ (1 \leq i \leq m)\}.$$

Hypotheses  $(H_1)$ ,  $(H_2)$  and  $(H_3)$  hold due to general facts of varieties defined by symmetric polynomials (see [MPP15] for details). Further, it can be shown that  $(H_4)$  holds by a generalization of the arguments proving the validity of  $(H_4)$  for the linear family  $\mathcal{A}_{\mathcal{L}}$ . As a consequence, applying Theorem 6.5 we deduce the following result.

**Theorem 6.7.** *For  $p := \text{char}(\mathbb{F}_q)$  not dividing  $d(d-1)$ ,  $m \leq s \leq d-m-4$  and  $q > \max\{d, 16(D_{\mathcal{N}}\delta_{\mathcal{N}} + 14D_{\mathcal{N}}^2\delta_{\mathcal{N}}^2q^{-\frac{1}{2}})^2\}$ , where  $\delta_{\mathcal{N}} := \prod_{i=1}^m d_i$  and  $D_{\mathcal{N}} := \sum_{i=1}^m (d_i - 1)$ , the following estimate holds:*

$$|\mathcal{V}(\mathcal{A}_{\mathcal{N}}) - \mu_d q| \leq 2^d \delta_{\mathcal{N}} (3D_{\mathcal{N}} + d^2) q^{\frac{1}{2}} + 67 \delta_{\mathcal{N}}^2 (D_{\mathcal{N}} + 2)^2 d^{d+5} e^{2\sqrt{d}-d}.$$

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